Study Guide – STAT 543, Exam 2

Below is a rough outline of the material covered on the second exam. This corresponds to material in the textbook from Chapter 6 and Chapter 8. For practice, you should check out problems from the textbook on these Sections.

1. Data Reduction Principles
   (a) Sufficiency: distribution of \( X = (X_1, X_2, \ldots, X_n) \) given \( S \) doesn’t depend on \( \theta \)
      i. Factorization Theorem (for determining sufficiency)
      ii. 1-1 functions of sufficient statistics are sufficient
   (b) Minimal sufficiency
      i. Theorem (for determining minimal sufficiency): Let \( f(x|\theta) \) be the joint pdf/pmf of \( X_1, \ldots, X_n, \ \theta \in \Theta \subset \mathbb{R}^p \). Let \( T = (T_1, \ldots, T_k) \) be a statistic such that: for every two sample points \( x \) and \( y \), \( \{ \theta : f(x|\theta) > 0 \} = \{ \theta : f(y|\theta) > 0 \} \) and \( f(x|\theta)/f(y|\theta) \) is constant over \( \theta \in \{ \theta : f(x|\theta) > 0 \} \) if and only \( T(x) = T(y) \). Then, \( T \) is minimal sufficient for \( \theta \).
      ii. Minimal sufficient statistics from exponential family (see below)
   (c) Completeness of a statistic \( T \) (or equivalently the family \( \{ f_T(t|\theta) : \theta \in \Theta \} \) of distributions)
      i. Sufficiency & (bounded) completeness together imply minimal sufficiency
   (d) Exponential families: definition and theorem for finding complete, sufficient statistics
   (e) Ancillary statistics
      i. Basu’s Theorem: If \( S \) is a complete, sufficient statistic & \( T \) is an ancillary statistic, then \( S, T \) are independent (no matter the data generating \( \theta \))

2. Sufficiency and UMVUEs
   (a) Rao-Blackwell Theorem (conditional expectation \( T^* = E(T|S) \) of an UE \( T \) of \( \gamma(\theta) \) given a sufficient statistic \( S \) is an unbiased estimator \( E_{\theta}\gamma(\theta) \) with \( \text{Var}_\theta(T^*) \leq \text{Var}_\theta(T) \) for all \( \theta \); and if \( \text{Var}_{\theta_0}(T^*) = \text{Var}_{\theta_0}(T) \) at some \( \theta_0 \), then it must be that \( P_{\theta_0}(T = T^*) = 1 \).
   (b) Lehmann-Scheffe Theorem (getting the UMVUE based on a complete, sufficient statistic)
      i. Method I for getting UMVUE of \( \gamma(\theta) \): Find an UE of \( \gamma(\theta) \) that is a function of a complete, sufficient statistic
      ii. Method II for getting UMVUE of \( \gamma(\theta) \): start with UE \( T \) of \( \gamma(\theta) \) & then UMVUE is \( E(T|S) \) where \( S \) is complete, sufficient

3. Testing of Hypotheses
   (a) Definitions:
      i. General notations: Null/alternative hypothesis, simple/composite hypothesis, test function/rule, simple test function, rejection region, acceptance region
      ii. Probability related: Type I, Type II errors, power function (power), size or level of test
   (b) Most Powerful (MP) tests of \( H_0 : \theta = \theta_0 \) vs \( H_1 : \theta = \theta_1 \)
      i. Definition of MP tests & Neyman-Pearson lemma for finding MP tests, i.e., a MP size \( \alpha \) test for \( H_0 : \theta = \theta_0 \) vs \( H_1 : \theta = \theta_1 \), is given by
         \[
         \varphi(x) = \begin{cases} 
         1 & \text{if } f(x|\theta_1) > kf(x|\theta_0) \\
         \gamma & \text{if } f(x|\theta_1) = kf(x|\theta_0) \\
         0 & \text{if } f(x|\theta_1) < kf(x|\theta_0) 
         \end{cases}
         \]
      where \( \gamma \in [0, 1] \) and \( 0 \leq k \leq \infty \) are constants satisfying \( E_{\theta_0}\varphi(X) = \alpha \)
(c) Two Methods for finding: Uniformly Most Powerful (UMP) size $\alpha$ tests of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$

i. Based on Neyman-Pearson lemma: Suppose you can pick some $\theta_0 \in \Theta_0$ so that 1) the MP test $\varphi(x)$ by the Neyman-Pearson lemma of $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ is the same, no matter what alternative $\theta_1$ you choose; and 2) the size of $\varphi(x)$ is $\alpha$, i.e., $\max_{\theta \in \Theta_0} \mathbb{E}_{\theta} \varphi(X) = \alpha$. Then $\varphi(x)$ is UMP of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$

ii. Using Monotone Likelihood Ratio property (in a real-valued statistic $T = t(X)$): UMP tests for ‘$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$’ or ‘$H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$’ where $\theta$ is real-valued in terms of $T = t(X)$

(d) Likelihood ratio statistics and tests for $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$

i. reject $H_0$ if $\lambda(x) = \frac{\max_{\theta \in \Theta_0} f(x|\theta)}{\max_{\theta \in \Theta} f(x|\theta)}$ is too small, i.e., the likelihood ratio test is

$$\varphi(x) = \begin{cases} 1 & \text{if } \lambda(x) < k \\ \gamma & \text{if } \lambda(x) = k \\ 0 & \text{if } \lambda(x) > k \end{cases}$$

where $\gamma \in [0, 1]$ and $0 \leq k \leq 1$ are constants determined by $\max_{\theta \in \Theta_0} \mathbb{E}_{\theta} \varphi(X) = \alpha$.

ii. Large sample properties of likelihood ratio test for hypotheses “$H_0 : \theta_i = \theta_i^0, \ldots, \theta_r = \theta_r^0$ vs $H_1 : \theta_i \neq \theta_i^0$ for some $1 \leq i \leq r$” (where $r \leq p$ and parameter $\theta = (\theta_1, \ldots, \theta_p)$ has $p$ components), i.e., $-2 \log \lambda(x)$ has a large sample $\chi^2(r)$ distribution for calibrating tests

(e) Bayes tests of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ based on prior pdf $\pi(\theta)$, i.e., reject if posterior probability $P(\theta \notin \Theta_0|x) = \int_{\Theta \setminus \Theta_0} f_{\theta|x}(\theta)d\theta \geq 1/2$