A final note on Bayes and Minimax Estimators

We have learned that minimax and Bayes principles are two ways of finding estimators based on risk functions. There are also some interesting connections between the two types of estimators. As an example, the following result shows how to find a minimax estimator (which is hard) from a Bayes estimator (which can be done more easily).

**Theorem:** For some loss function \( L(t, \theta) \), if \( T^* \) is a Bayes estimator with respect to some prior and the risk of \( T^* \) is constant (i.e., \( R_{T^*}(\theta) = c \) for all \( \theta \in \Theta \)), then \( T^* \) is the minimax estimator under the same loss function.\(^1\)

**Example:** For \( X_1, \ldots, X_n \) iid Bernoulli(\( \theta \)), \( \theta \in \Theta = (0, 1) \), find the minimax estimator of \( \theta \) under the loss function \( L(t, \theta) = (\theta - t)^2 / \{\theta(1 - \theta)\} \).

**Solution:** For this loss function, the Bayes estimator of \( \theta \) with respect to the uniform(0,1) prior on \( \Theta \) is given by \( T_0 = \bar{X}_n \) (see Homework 3). Also note that, for any \( \theta \in (0, 1) \),

\[
R_{T_0}(\theta) = \mathbb{E}_\theta \left( \frac{(\theta - \bar{X}_n)^2}{\theta(1 - \theta)} \right) = \frac{1}{\theta(1 - \theta)} \text{MSE}_\theta(\bar{X}_n)
\]

\[
= \frac{1}{\theta(1 - \theta)} \text{Var}_\theta(\bar{X}_n) \quad \text{since } \bar{X}_n \text{ is unbiased}
\]

\[
= \frac{1}{n} \quad \text{since } \text{Var}_\theta(X_1) = \theta(1 - \theta) = n \text{Var}_\theta(\bar{X}_n)
\]

Hence, \( T_0 \) has constant risk so that, by the theorem, \( T_0 = \bar{X}_n \) is also the minimax estimator of \( \theta \).

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\(^1\)Proof: Suppose that \( T^* \) is a Bayes estimator with respect to a prior pdf \( \pi(\theta) \) on \( \Theta \). Then the Bayes risk of \( T^* \) is \( BR_{T^*} = \int_{\Theta} R_{T^*}(\theta)\pi(\theta)d\theta = c \int_{\Theta} \pi(\theta)d\theta = c \), using that \( R_{T^*}(\theta) = c \) is constant and \( \int_{\Theta} \pi(\theta)d\theta = 1 \). Since \( T^* \) is the Bayes estimator, it has minimal Bayes risk by definition so that, for any other estimator \( T \), we have

\[
c = BR_{T^*} \leq BR_T = \int_{\Theta} R_T(\theta)\pi(\theta)d\theta \leq \int_{\Theta} \left[ \max_{\theta \in \Theta} R_T(\theta) \right] \pi(\theta)d\theta = \left[ \max_{\theta \in \Theta} R_T(\theta) \right] \int_{\Theta} \pi(\theta)d\theta = \max_{\theta \in \Theta} R_T(\theta)
\]

or \( c \leq \max_{\theta \in \Theta} R_T(\theta) \) for any estimator \( T \). So it must be that

\[
\max_{\theta \in \Theta} R_T(\theta) = c = \min_T \max_{\theta \in \Theta} R_T(\theta),
\]

implying that \( T^* \) is minimax.