3.9 Solving another difference equation

In an earlier theorem, we saw that the solution to

$$F(t+1) = R\,F(t)$$

is $F(t) = F(0)\,R^t$.
Here we study a different difference equation.

Theorem 3.9.1. Consider the difference equation
$F(t+1) = R\,F(t) + K$
where R and K are constant.
(1) If $R \neq 1$, then every solution has the form

$$F(t) = C\,R^t + E$$

where E is the equilibrium $E = K / (1-R)$
for some constant C.
(2) If R = 1 then every solution has the form

$$F(t) = C + K\,t$$

for some constant C.

This theorem is analogous to the earlier theorem that the solution to $F(t+1) = R\,F(t)$ is $F(t) = F(0)\,R^t$.

Example. Solve $F(t+1)= 1.3\,F(t) - 150$
if $F(0) = 2000$.

Solution.
Note that the equation is in the form

$$F(t+1) = R\,F(t) + K$$

where $R = 1.3$ and $K = -150$.
Part (1) of the theorem applies since $R \neq 1$.
By the theorem, to solve it we must find the equilibrium E.

$$E = K / (1-R)$$
$$= -150 / (1-1.3)$$
$$= 150/0.3 = 500$$

By the theorem, the solution is

$$F(t) = C\,R^t + E.$$

Hence $F(t) = C\,(1.3)^t + 500$
But $F(0) = 2000$, so
$C + 500 = 2000$
$C = 1500$
Hence $F(t) = 1500\,(1.3)^t + 500$.

Note $C \neq F(0)$.

We call the formula $F(t) = C\,(1.3)^t + 500$ the **general solution** to the problem, since every solution has that form. Often a problem breaks into steps:
Step 1: Find the general solution. This will contain an unknown constant.
Step 2: Use the initial condition to evaluate the unknown constant.

Case (1) in the theorem is most common since, most often, $R\neq1$. But the special case R = 1 is also useful.

Example. Solve $F(t+1)= F(t) +5$

if F(0) = 2000.
Solution. Here R = 1 and K = 5. Part (2) of the theorem applies to say
F(t) = C + 5 t.
Since F(0) = C + 5 (0) = 2000, we find C = 2000.
The answer is then
F(t) = 2000 + 5 t.

We can easily check that this formula is a solution to this equation: Note
F(t+1) = 2000 + 5 (t+1)
so we need only verify
2000 + 5 (t+1) = 2000 + 5 t + 5
5t+5 = 5t + 5.
It checks.

Example. Suppose P(t+1)-P(t) = -0.2 P(t) + 400.
(a) Find the general solution.
(b) Find the solution if P(0) = 750
(c) Check the solution.
(d) If P(0) = 750, find P(5).

Solution.
(a) The equation can be rewritten in the form
P(t+1) = R P(t) + K
as

P(t+1) = P(t) - 0.2 P(t) + 400
P(t+1) = 0.8 P(t) + 400.

Because of its form, we seek the equilibrium E.
The equilibrium E satisfies
E = 0.8 E + 400
0.2 E = 400
E = 400/0.2 = 2000
Hence the general solution is C R^t + E or
P(t) = C $(0.8)^t$ + 2000

(b) We now utilize the initial information to find C:
P(0) = C + 2000 = 750
C = -1250
Hence the solution is P(t) = -1250 $(0.8)^t$ + 2000.

(c) To check the solution we plug the solution into the equation

P(t+1) = 0.8 P(t) + 400

and use algebra to verify that the relationship holds.
Thus we ask whether
-1250 $(0.8)^{t+1}$ + 2000 = 0.8 [-1250 $(0.8)^t$ + 2000] + 400
is true. To verify it, we simplify.
-1250 $(0.8)^{t+1}$ + 2000 = 0.8 [-1250 $(0.8)^t$] + 0.8 [2000] + 400
-1250 $(0.8)^{t+1}$ + 2000 = 0.8 [-1250 $(0.8)^t$] + 1600 + 400
-1250 $(0.8)^{t+1}$ + 2000 = 0.8 [-1250 $(0.8)^t$] + 2000
Cancel the 2000:
-1250 $(0.8)^{t+1}$ + 2000 = 0.8 [-1250 $(0.8)^t$] + 2000
Rewrite $(0.8)^{t+1}$ = (0.8) $(0.8)^t$

$-1250\ (0.8)\ (0.8)^t = 0.8\ [\ -1250\ (0.8)^t\ ]$
We see that it checks.

(d) Now $P(5) = -1250\ (0.8)^5 + 2000 = 1590$

Example. (a) Solve $P(t+1) - P(t) = -0.1\ P(t) + 200$
if $P(0) = 1000$
(b) Find $P(10)$.
(c) Estimate t such that $P(t) = 1800$. Give your answer to three significant figures.

Solution. (a) The equilibrium E is
$E - E = -0.1\ E + 200$
$0.1\ E = 200$
$E = 2000$
The equation becomes $P(t+1) = 0.9\ P(t) + 200$,
which has the form $P(t+1) = R\ P(t) + K$ with $R = 0.9$.
Hence $P(t) = C\ (0.9)^t + 2000$. This is the general solution.
When $t = 0$, $P(0) = C + 2000 = 1000$.
Hence $C = -1000$.
The answer is:
$$P(t) = 2000 - 1000\ (0.9)^t$$

(b) $P(10) = 2000 - 1000(0.9)^{10} = 1651$
(c) Solve $P(t) = 1800$
$2000 - 1000\ (0.9)^t = 1800$
$-1000\ (0.9)^t = -200$
$10\ (0.9)^t = 2$
$\log 10 + t \log(0.9) = \log 2$
$t = (\log 2 - \log 10)/\log(0.9) = 15.3$

Example. Suppose $P(t+1)-P(t) = 0.2\ P(t) - 200$.
(a) Find the general solution.
(b) Find the solution if $P(0) = 750$
(c) Check the solution of (b).
(d) If $P(0) = 750$, find $P(5)$.

Solution.
(a) The equilibrium E satisfies
$E-E = 0.2\ E - 200$
$0.2\ E = 200$
$E = 1000$
The form of the equation is $P(t+1) = 1.2\ P(t) - 200$.
Hence the general solution is
$P(t) = C\ (1.2)^t + 1000$

(b) We now utilize the initial information to find C:
$P(0) = C + 1000 = 750$
$C = -250$
Hence the solution is $P(t) = -250\ (1.2)^t + 1000$.

(c) Indeed $P(0) = 750$.

Is P(t+1) - P(t) = 0.2 P(t) - 200?
Is P(t+1) = 1.2 P(t) - 200?
Is -250 $(1.2)^{t+1}$ +1000 = 1.2(-250 $(1.2)^{t}$ + 1000) -200?
Is -250$(1.2)^{t+1}$ + 1000 = (1.2)(-250)$(1.2)^{t}$ + 1.2(1000) - 200?
Is -250$(1.2)^{t+1}$+1000 = (-250)$(1.2)^{t+1}$ + 1200 - 200?
Yes.

(d) P(5) = -250 $(1.2)^{5}$ + 1000 = 377.92

We use Theorem 3.9.1 so much that it is important to know why it is true.
Consider the difference equation

F(t+1) = R F(t) + K

where K is a constant.
For case (1), we assume R ≠ 1.
Note that the equilibrium E = K / (1-R) solves the equation

E = R E + K

Write G(t) = F(t) - E so F(t) = G(t) + E
Then G(t+1) = F(t+1) - E so F(t+1) = G(t+1) + E.
Plug into the equation:

F(t+1) = R F(t) + K
G(t+1) + E = R (G(t) + E) + K
G(t+1) + E = R G(t) + RE + K

But since E is an equilibrium,

E = RE + K

Subtracting,

G(t+1) = R G(t)

This equation is the familiar one from section 3.2, which we already know how to solve.
Hence

G(t) = G(0) R^{t}

But F(t) = G(t) + E
so

F(t) = G(0) R^{t} + E.
F(t) = C R^{t} + E.

Here the constant would be C = G(0). It is usually easiest just to solve for C when it is needed.

For case (2), we know R = 1. In this case there is no equilibrium E and a different approach is required. Observe that the equation is

F(t+1) = F(t) + K

Hence F(1) = F(0) + K
F(2) = F(1) + K = (F(0) + K) + K = F(0) + 2 K
F(3) = F(2) + K = (F(0) + 2 K) + K = F(0) + 3 K
F(4) = F(3) + K = (F(0) + 3 K) + K = F(0) + 4 K.
The pattern is now clear:
F(t) = F(0) + t K.
This has the form F(t) = C + K t with C = F(0).