5.6 The chain rule

The **composition** $g \circ f$ of the functions f and g is the function
$(g \circ f)(x) = g(f(x))$.
This means, "do the function f to x, then do g to the result."

Example. $g(x) = x^2$ and $f(x) = (3x+1)$.
Then $(g \circ f)(x) = g(f(x)) = g(3x+1) = (3x+1)^2$.
But $(f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 + 1$.

Just as in this example, usually $g \circ f$ and $f \circ g$ are not the same.

Example. Write $(2x+3)^4$ as $g \circ f$, by telling what is g and what is f.
Solution. In $g \circ f$, g tells the last thing done. So $g(x) = x^4$. Simultaneously, f tells what g is done to, so $f(x) = 2x+3$.

Example. Write $\sqrt{(5x^3 + 7)}$ as $g \circ f$.
Solution. $g(x) = \sqrt{x}$ since the last thing done is to take the square root. And
$f(x) = 5x^3 + 7$.

The main result for differentiation of compositions of functions is called the "Chain Rule."

Chain Rule. Suppose f and g have derivatives. Then
$$(g \circ f)'(x) = g'(f(x))\, f\,'(x).$$

We will see this in many guises. Following is the result where $g(u) = u^n$, so
$(g \circ f)(x) = (f(x))^n$.

Example. If $k(x) = (3x^2+1)^{10}$, find $k'(x)$
Solution. We recognize that $k(x) = (g \circ f)(x)$ where $g(x)= x^{10}$ and $f(x) = (3x^2 + 1)$.
Hence $g\,'(x) = 10\, x^9$ and $f\,'(x) = 6\, x$.

By the Chain Rule
$k\,'(x) = (g \circ f)'(x) = g'(f(x))\, f\,'(x)$
$= 10\, (3x^2+1)^9\, (6x)$
$= 60\, x(3x^2+1)^9$.

This kind of example occurs so often that there is an explicit formula for differentiating a power:

Corollary. (Power Chain Rule) For any integer,
$$D_x[\, [f(x)]^n] = n\, [f(x)]^{n-1}\, D_x[f(x)]$$

In words, to find the derivative of an expression to any power, find
old power * expression to one lower power * derivative of the expression

We will now redo the prvious example, using the Power Chain Rule rather than the Chain Rule. The Power Chain Rule is a bit shorter and easier, but also one more thing to remember.

Example. If $k(x) = (3x^2+1)^{10}$, find $k'(x)$
Solution. By the Power Chain Rule with n = 10, $f(x) = (3x^2+1)$, we find
$k'(x) = 10\,(3x^2+1)^9\, D_x(3x^2+1)$
$= 10\,(3x^2+1)^9\,(6x)$
$= 60\,x(3x^2+1)^9$

Example. Let $g(x) = (x^3 + 2)^4$
(a) Find g'(x).
(b) Find the equation of the line tangent to the curve y = g(x) when x = 1.
Solutions.
(a) By the Power Chain Rule,
$g'(x) = 4\,(x^2+2)^3\,(3\,x^2)$
$= 12\,x^2\,(x^2+2)^3$

(b) When x = 1, $y = g(1) = (1^3+2)^4 = 3^4 = 81$.
the slope is $g'(1) = 12(1)(1^2+2)^3 = 12(3)^3 = 324$.
Hence the line is $y = y_1 + m(x-x_1)$
y = 81 + 324(x-1)
y = 81 + 324x - 324
y = 324 x - 243

Example. Find $d/dt\,[\,3/(t^2+1)^4]$
Solution. $d/dt\,[\,3/(t^2+1)^4]$ is the derivative of $3\,(t^2+1)^{-4}$
$= d/dt\,[\,3(t^2+1)^{-4}]$
$= 3\,(-4)\,(t^2+1)^{-5}\,D_t(t^2+1)$
$=-12\,(t^2+1)^{-5}\,(2t)$
$=-24\,t\,(t^2+1)^{-5}$

Example. Let $f(x) = (2\,x^3 + 3)^7$
(a) Find the derivative of f.
(b) Find the instantaneous rate of change when x = 1
Solution.
(a) $f'(x) = 7\,(2\,x^3 + 3)^{7-1}\,D_x(2\,x^3 + 3)$
$= 7\,(2\,x^3 + 3)^6\,(6\,x^2\,)$
$= 42\,x^2\,(2\,x^3 + 3)^6$
(b) The instantaneous rate of change is f '(1) = 656,250.

Example. If $f(t) = 7\,(3\,t^4 - t)^5$, find f '(t).
Solution.
$f'(t) = 7\,(5)\,(3\,t^4 - t)^4\,D_t(3\,t^4 - t)$
$= 35\,(3\,t^4 - t)^4\,(12\,t^3 - 1)$

We can combine the Power Chain Rule with other rules:

Example. Find $D_x\,[\,x^2\,(x^3 + 2)^5]$.
Solution. Since the expression is a product, we first use the product rule:

$D_x [x^2 (x^3 + 2)^5]$.
$= x^2 D_x [(x^3 + 2)^5].+ (x^3 + 2)^5 D_x [x^2]$.
$= x^2 (5)(x^3 + 2)^4 D_x[x^3+2].+ (x^3 + 2)^5 (2x)$
$= 5x^2 (x^3 + 2)^4 (3x^2).+ (x^3 + 2)^5 (2x)$
$= 15x^4 (x^3 + 2)^4 .+ 2x(x^3 + 2)^5$

Example. Find the derivative of $(x^2 + 1)^4 (2x^3)$
Solution.
$(x^2 + 1)^4 D_x(2x^3) + (2x^3) D_x[(x^2 + 1)^4]$
$= (x^2 + 1)^4 (6x^2) + (2x^3) 4 (x^2 + 1)^3 D_x[(x^2 + 1)]$
$= (x^2 + 1)^4 (6x^2) + (2x^3) 4 (x^2 + 1)^3 (2x)$
$= (x^2 + 1)^4 (6x^2) + (16x^4) (x^2 + 1)^3$

Example. Find the derivative of $\dfrac{x^2}{(x^2+2)^3}$

Solution

This is $\dfrac{(x^2+2)^3 D_x (x^2) - x^2 D_x[(x^2+2)^3]}{[(x^2+2)^3]^2}$

$= \dfrac{(x^2+2)^3 (2x) - x^2 3(x^2+2)^2 D_x[x^2+2]}{[(x^2+2)^3]^2}$

$= \dfrac{(x^2+2)^3 (2x) - x^2 3(x^2+2)^2(2x)}{(x^2+2)^6}$

$= \dfrac{(x^2+2)^3 (2x) - 6x^3 (x^2+2)^2}{(x^2+2)^6}$

$= \dfrac{(x^2+2)^2[(x^2+2) (2x) - 6x^3]}{(x^2+2)^6}$

$= \dfrac{[(x^2+2) (2x) - 6x^3]}{(x^2+2)^4}$

$= \dfrac{[2x^3+4x - 6x^3]}{(x^2+2)^4}$

$$= \frac{-4x^3+4x}{(x^2+2)^4}$$

Example. Find $D_x [(x^3+2)^4 (x^2+3)^5]$
Solution. We use the product rule:
$D_x [(x^3+2)^4 (x^2+3)^5]$
$= (x^3+2)^4 D_x [(x^2+3)^5]+ (x^2+3)^5 D_x [(x^3+2)^4]$
$= (x^3+2)^4\ 5\ (x^2+3)^4 D_x [x^2+3]+ (x^2+3)^5\ 4(x^3+2)^3 D_x [x^3+2]$
$= (x^3+2)^4\ 5\ (x^2+3)^4 (2x)+ (x^2+3)^5\ 4(x^3+2)^3 (3x^2)$
$= 10x(x^3+2)^4 (x^2+3)^4 + 12 x^2 (x^2+3)^5 (x^3+2)^3$

It is not always easy to decide what is the simplest form for the answer, since what is simplest depends on the use that is to be made of the formula. In this case, we can factor out x from both terms, as well as $(x^3+2)^3$ since both terms contain that many powers of $(x^3 + 2)$, and $(x^2 + 3)^4$ since both terms contain that many powers of $(x^2 + 3)$. We obtain

$10x(x^3+2)^4 (x^2+3)^4 + 12 x^2 (x^2+3)^5 (x^3+2)^3$
$= x (x^3+2)^3 (x^2 + 3)^4 [10 (x^3 + 2) + 12 x (x^2+3)]$
$= x (x^3+2)^3 (x^2 + 3)^4 [10 x^3 + 20 + 12 x^3+36 x]$
$= x (x^3+2)^3 (x^2 + 3)^4 [22 x^3 +36 x + 20]$
which is the best form for the answer.

The use of the Power Chain Rule also leads to the following formula:

Theorem $D_x [x^n] = n x^{n-1}$ for all constants n.

Example. $D_x [x^{5/7}]$
$= (5/7) x^{(5/7)-1}$
$= (5/7) x^{-(2/7)}$

Expressions written with surds are just powers written in a funny way. To find their derivatives, rewrite them in terms of their powers.

Example. $D_x [\sqrt{x}]$
$= D_x [x^{1/2}]$
$= (1/2) x^{-1/2}$

Example. D_x [cube root of x^2]
$= D_x [x^{2/3}]$
$= (2/3) x^{(2/3)-1}$
$= (2/3) x^{(-1/3)}$

The power chain rule also works in this setting with constant exponents

Theorem (Power Chain Rule for any constant exponent) For any constant n,

$D_x[\ [f(x)]^n] = n\ [f(x)]^{n-1}\ D_x[f(x)]$

Example. $D_x\ [\ \sqrt{(x^3+4)}]$
$= D_x\ [(x^3+4)^{(1/2)}\ ]$
$= (1/2)\ (x^3+4)^{(1/2)-1}\ D_x(x^3+4)$
$= (1/2)\ (x^3+4)^{(-1/2)}\ 3x^2$
$= (3/2)x^2\ (x^3+4)^{(-1/2)}$

Example. $D_x\ [\ x(x^2+1)^{1/4}]$
$=\ x\ D_x\ [\ (x^2+1)^{1/4}] +\ (x^2+1)^{1/4}\ D_x[x]$
$=\ x\ (1/4)\ (x^2+1)^{(1/4)-1} D_x\ [\ x^2+1] +\ (x^2+1)^{1/4}$
$=\ x\ (1/4)\ (x^2+1)^{(-3/4)}(2x) +\ (x^2+1)^{1/4}$
$=\ (x^2/2)\ (x^2+1)^{(-3/4)} +\ (x^2+1)^{1/4}$

Example. $D_t\ [\ 4\ t\ \sqrt{(t^3)}\ ]$
$=\ D_t\ [\ 4\ t\ (t^{3/2})\ ]$
$=\ D_t\ [\ 4\ (t^{5/2})\ ]$
$= 4\ (5/2)\ t^{(5/2)-1}$
$= 10\ t^{3/2}$

Problems for 5.6

1. If $f(x) = (2x^2 + 3)^8$, find $f\ '(x)$.
Ans: $32\ x\ (2x^2+3)^7$

2. If $g(x) = (3x^3 + 2x + 1)^6$, find $g'(x)$.
Ans: $6\ (3x^3 + 2x + 1)^5\ (9\ x^2 + 2)$

3. Find $D_x\ [\ (x^3 + 1)^5]$
Ans: $15\ x^2\ (x^3 + 1)^4$

4. Suppose $P(t) = (t^4 + 1)^4$
(a) Find $P'(t)$.
(b) Find the equation of the line tangent to $y = P(t)$ at $t = 1$.
(c) Find the instantaneous rate of change when $t = 2$.
Ans:
(a) $16\ t^3\ (t^4+1)^3$
(b) $y = 128\ t - 112$
(c) 628,864

5. Suppose $f\ (x) = x^3\ (x^2 + 1)^4$
(a) Find $f\ '(x)$.
(b) Find $f\ '(1)$.
(c) Find the equation of the line tangent to $y = f(x)$ at $x = 1$.
Ans:
(a) $8\ x^4\ (x^2 + 1)^3 + 3\ x^3\ (x^2 + 1)^4$
(b) 112

(c) $y = 112x - 96$

6. Suppose $g(x) = 2x^2 (3x^3 + 1)^3$
(a) Find g'(x).
(b) Find g'(1).
Ans:
(a) $54x^4 (3x^3 + 1)^2 + 4x (3x^3 + 1)^3$
(b) 1120

7. Find the derivative:
(a) $(x^3 - 4x^2 + 5)^5$
(b) $1/(2t^2 - 3t)^4$
(c) $2x^2 (3x+7)^4$
(d) $\sqrt{(x^4 + 3)}$
(e) $5x/(2x+1)^3$
(f) $(x^2 + 3)^{3/4}$
Ans: (a) $5 (x^3 - 4x^2 + 5)^4 (3x^2 - 8x)$
(b) $-4 (2t^2 - 3t)^{-5} (4t - 3)$
(c) $24x^2 (3x+7)^3 + 4x (3x+7)^4$
(d) $2x^3 (x^4 + 3)^{-1/2}$
(e) $(5 - 20x) / (2x+1)^4$
(f) $(3x/2) (x^2 + 3)^{-1/4}$

8. Find the equation of the line tangent to the graph of $y = 3/(x-2)^2$ at $x = 1$.
Ans: $y = 6x-3$

9. Suppose $g(x) = \dfrac{2x+1}{(3x+1)^3}$
(a) Find g'(x).
(b) Find g'(1).
Ans:
(a) $-\dfrac{12x + 7}{(3x+1)^4}$
(b) $-19/256$

10. Suppose $f(x) = \dfrac{2x + 1}{(x-2)^2}$
(a) Find f '(x).
(b) Find f '(1).
Ans:
(a) $-\dfrac{2x + 6}{(x-2)^3}$
(b) 8

11. Suppose $g(x) = \dfrac{2x^3}{(x^2 + 1)^4}$

(a) Find g'(x).
(b) Find the slope of the line tangent to y = g(x) at x = 2.
Ans:
(a) $\dfrac{x^2 (6 - 10x^2)}{(x^2 + 1)^5}$

(b) - 136 / 3125

12. Suppose $g(t) = 2t^2 + 3t + 3\sqrt{t}$
(a) Find g'(t)
(b) Find g'(4).
Ans:
(a) $4t + 3 + (3/2)\, t^{-1/2}$
(b) 19.75

13. Suppose $f(x) = (2x^3 + 3)^{-4}$.
(a) Find f ' (x).
(b) Find the slope of the line tangent to y = f(x) where x = 1.
Ans: (a) $-24x^2 (2x^3+3)^{-5}$
(b) -0.00768

14. Suppose $g(x) = (3x^2 + 1)^{-(2/3)}$.
(a) Find g ' (x).
(b) Find the slope of the line tangent to y = g(x) where x = 2.
Ans: (a) $-4x (3x^2 + 1)^{-(5/3)}$
(b) -0.1113058

15. Suppose $g(x) = (4x - 3)^{-(1/2)}$.
(a) Find g ' (x).
(b) Find the slope of the line tangent to y = g(x) where x = 1.
Ans: (a) $-2 (4x - 3)^{-(3/2)}$
(b) -2

16. Suppose $f(x) = \sqrt{(x^3 +1)}$
Find f '(x).
Ans: $(3/2)\, x^2 (x^3 + 1)^{-1/2}$

$= \dfrac{3x^2}{2\sqrt{(x^3 + 1)}}$

17. Suppose $f(x) = (x^3 +1)^{4/3}$
Find f '(x).
Ans: $4x^2 (x^3 + 1)^{1/3}$

18. Suppose $f(x) = x\sqrt{(2x^2 +1)}$
Find f '(x).
Ans: $2x^2 (2x^2 + 1)^{-(1/2)} + (2x^2 + 1)^{(1/2)}$

$$= \frac{4x^2 + 1}{\sqrt{(2x^2 + 1)}}$$

19. Suppose $g(x) = 2x^{2/3} + 3x^{4/3}$
(a) Find g'(x)
(b) Find g'(8).
Ans:
(a) $(4/3)\, x^{-(1/3)} + 4\, x^{(1/3)}$
(b) 26/3

20. Suppose $g(x) = 3x^{(1/3)} + 6x^{(2/3)}$
(a) Find g'(x).
(b) Find g'(8).
(c) Find the equation of the line tangent to y = g(x) at x = 8.
Ans:
(a) $x^{-(2/3)} + 4x^{-(1/3)}$
(b) 9/4 = 2.25
(c) y = 2.25 x + 25.5

21. Suppose $g(x) = x^4 (x^2 + 3x + 1)^6$
(a) Find g'(x).
(b) Find g'(1).
Ans: (a) $6x^4 (x^2 + 3x + 1)^5 (2x + 3) + 4x^3 (x^2 + 3x + 1)^6$
(b) 156,250

22. Suppose $g(x) = (x^2 + 1)^3 (x^2 + 2x + 3)^4$
Find g'(x).
Ans: $8(x+1)(x^2 + 1)^3 (x^2 + 2x + 3)^3 + 6x(x^2 + 1)^2 (x^2\ \ 2x + 3)^4$

23. Suppose $g(x) = \dfrac{(2x+1)^2}{(x^2 + 2x + 3)^3}$
Find g'(x).
Ans: $\dfrac{(2x+1)(6 - 10x - 8x^2)}{(x^2 + 2x + 3)^4}$

24. Suppose $g(x) = \dfrac{\sqrt{(2x+1)}}{\sqrt{(3x + 2)}}$
(a) Find g'(x).
(b) Find g'(2)
Ans: (a) $\dfrac{1}{2(3x + 2)^{3/2} (2x+1)^{1/2}}$
(b) 0.02582

25. Suppose $P(t) = t^2\, t^{(3/5)}$.
(a) Find P'(t).
(b) Find the instantaneous rate of change of P when t = 32.

Ans:
(a) $(13/5)\, t^{(8/5)}$
(b) 3328/5 = 665.6

26. Suppose g(x) = $\dfrac{(3x + 2)}{(x^2 + 4x + 1)^2}$

Find g'(x).
Ans:

$$- \frac{9 x^2 + 20 x + 13}{(x^2 + 4 x + 1)^3}$$

27. Suppose P(t) = $\dfrac{6 t^2}{t^{(2/3)}}$.

(a) Find P'(t).
(b) Find the instantaneous rate of change of P when t = 8.
Ans: (a) $8 t^{(1/3)}$ (b) 16