1. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and $P$ is a partition of $[a, b]$. Show that there exists a Riemann sum of $f$ over $P$ that equals $\int_a^b f(x) dx$.

**Solution:** Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$. Denote $\Delta_i = x_{i+1} - x_i$, $m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$, and $M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$.

By the intermediate value theorem for integrals,

$$m_i \Delta_i \leq \int_{x_i}^{x_{i+1}} f(x) dx \leq M_i \Delta_i, \quad i = 0, \ldots, n - 1.$$ 

Since $f$ is continuous, there exists $s_i, t_i \in [x_i, x_{i+1}]$ such that

$$m_i = f(s_i) \quad \text{and} \quad M_i = f(t_i), \quad i = 0, \ldots, n - 1.$$ 

Since

$$f(s_i) \leq \frac{1}{\Delta_i} \int_{x_i}^{x_{i+1}} f(x) dx \leq f(t_i)$$

and $f$ is continuous, there exist $r_i \in [x_i, x_{i+1}]$ such that $f(r_i) = \frac{1}{\Delta_i} \int_{x_i}^{x_{i+1}} f(x) dx$. Then, by the additivity property of integrals,

$$\sum_{i=0}^{n-1} f(r_i) \Delta_i = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) dx = \int_a^b f(x) dx,$$

as desired.

2. Compute $\sup_P L(P, f_i)$ and $\inf_P U(P, f_i)$ on $[0, 1]$ for $i = 1, 2$, and the following two functions defined on $[0, 1]$:

$$f_1(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}.$$

**Solution:** Both $\mathbb{Q}$ and its complement are dense in any interval of $\mathbb{R}$. Hence $L(P, f_1) = 0$ while $U(P, f_1) = 1$ for any partition $P$ of $[0, 1]$. Similarly, $U(P, f_2) = 1$ for any partition $P$. 

---

1
Observe next that for an arbitrary partition \( P = \{x_0 = 0, x_1, \ldots, x_{n-1}, x_n = 1\} , \)

\[ L(P, f_2) = \sum_{i=0}^{n-1} x_i(x_{i+1} - x_i) \]
is a Riemann sum of \( \int_0^1 x \, dx = 1/2 \). Thus \( \sup_P L(P, f_2) = \sup_P L(P, x) = \int_0^1 x \, dx = 1/2 \).

3 [20 Points].

(a) Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is absolutely integrable on \([0, +\infty)\) then

\[ \lim_{n \to \infty} \int_1^{\infty} f(x^n) \, dx = 0. \]

(b) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuously differentiable and one-to-one on \([a, b]\). Prove that

\[ \int_a^b f(x) \, dx + \int_{f(a)}^{f(b)} f^{-1}(x) \, dx = bf(b) - af(a). \]

where \( f^{-1} \) is the inverse function of \( f \).

Solution:

(a) Using change of variables \( x^n = t \) and the fact that

\[ \frac{dt}{dx} = nx^{n-1} = nt^{(n-1)/n}, \]

we obtain

\[ \left| \int_1^{\infty} f(x^n) \, dx \right| = \int_1^{\infty} |f(x^n)| \, dx \]

\[ = \frac{1}{n} \int_1^{\infty} |f(t)| t^{-\frac{n-1}{n}} \, dt \leq \frac{1}{n} \int_1^{\infty} |f(t)| \, dt < +\infty. \]

Hence \( \lim_{n \to \infty} \int_1^{\infty} f(x^n) \, dx = 0 \), by the squeeze theorem.

(b) Using change of variables \( x = f(t) \) and integrating by part, we obtain

\[ \int_{f(a)}^{f(b)} f^{-1}(x) \, dx = \int_a^b f^{-1}(f(t)) f'(t) \, dt = \int_a^b tf'(t) \, dt \]

\[ = tf(t) \big|_a^b - \int_a^b f(t) \, dt = bf(b) - af(a) - \int_a^b f(t) \, dt, \]

from which the claim follows.

Remark: The identity that we proved admits a simple geometric interpretation in terms of areas under the curves \( y = f(x) \) and \( y = f^{-1}(x) \). To figure it out, use the fact that the graph of \( f \) is the reflection of the graph of \( f^{-1} \) across the line \( y = x \).
4. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is bounded and \( f^2 \) is integrable on \([a, b]\). Does it follow that \( f \) is integrable on \([a, b]\)?

**Solution:** Let

\[
  f(x) = \begin{cases} 
    1 & \text{if } x \in \mathbb{Q} \\
    -1 & \text{if } x \not\in \mathbb{Q}
  \end{cases}
\]

Both the set of rational numbers and the set of irrational numbers are dense in any interval. Therefore, \( U(P, f) = 1 \) whereas \( L(P, f) = -1 \) for any partition \( P \) of \([a, b]\). Thus \( f \) is not integrable on \([0, 1]\) even though \( f^2 = 1 \) is.

5. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous and non-negative on \([0, 1]\). Show that if \( \int_0^1 f(x) = 0 \), then \( f(x) = 0 \) for every \( x \in [0, 1] \).

**Solution:** Since \( f \geq 0 \), we have \( \int_a^b f(x) dx \geq \int_c^d f(x) dx \) for any \( a \leq c \leq d \leq b \). However, if \( f(x_0) > 0 \) for some \( x_0 \in [a, b] \), then, since \( f \) is continuous, there exists \([c, d] \subset [a, b]\) such that \( x_0 \in [c, d] \) and \( f(x) \geq f(x_0)/2 \). Then

\[
  \int_a^b f(x) dx \geq \int_c^d f(x) dx \geq \frac{f(x_0) \cdot (d - c)}{2} > 0,
\]

which yields a contradiction.


**Solution:** Consider, for instance, \( f = 0 \) and

\[
  f_n(x) = \begin{cases} 
    0 & \text{if } x \in [a, b - 1/n] \\
    n^{3/2}(x - b + 1/n) & \text{if } x \in [b - 1/n, b]
  \end{cases}
\]

Then

\[
  \int_a^b |f_n(x) - f(x)| dx = n^{3/2} \int_0^{1/n} x dx = \frac{1}{2\sqrt{n}},
\]

and hence \( \lim_{n \to \infty} f_n = f \) in \( C_{\text{int}} \). On the other hand,

\[
  \max_{x \in [a, b]} |f_n(x) - f(x)| = |f_n(b) - f(b)| = \sqrt{n},
\]

and hence \( f_n - f \) is a divergent sequence in \( C_{\text{max}} \).