1. Let $\alpha$ and $c$ be real numbers, $c > 0$, and $f$ is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} 
  x^\alpha \sin(x^c) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}$$

Find all the values of the parameters $\alpha$ and $c$ such that:

(a) $f$ is continuous.

(b) $f'(0)$ exists.

(c) $f'$ is continuous.

(d) $f''(0)$ exists.

**Solution:** First, notice that the only “problematic” point is 0. Everywhere else, the function is differentiable infinitely many times. Thus:

(a) Since whenever $\lim_{x \to 0} f(x)$ exists we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left( x^{\alpha+c} \cdot \frac{\sin(x^c)}{x^c} \right) = \lim_{x \to 0} x^{\alpha+c},$$

then $\lim_{x \to 0} f(x) = 0$ if and only if $\alpha > -c$.

(b) We have:

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left( x^{\alpha-1+c} \cdot \frac{\sin(x^c)}{x^c} \right) = \lim_{x \to 0} x^{\alpha-1+c},$$

and hence $f'(0)$ exists if and only if $\alpha \geq 1 - c$. In fact,

$$f'(0) = \begin{cases} 
  0 & \text{if } \alpha > 1 - c \\
  1 & \text{if } \alpha = 1 - c.
\end{cases}$$

(1)

(c) Away from zero we have:

$$f'(x) = \alpha x^{\alpha-1} \sin(x^c) + cx^{\alpha+c-1} \cos(x^c) = x^{\alpha-1+c} \left( \alpha \cdot \frac{\sin(x^c)}{x^c} + c \cdot \cos(x^c) \right),$$

and hence

$$\lim_{x \to 0} f'(x) = \begin{cases} 
  0 & \text{if } \alpha > 1 - c \\
  \alpha + c = 1 & \text{if } \alpha = 1 - c \text{ and } c \geq 0.
\end{cases}$$

It then follows from (1) that $f'$ is continuous if and only if either $\alpha > 1 - c$ or $\alpha = 1 - c$ and $c \geq 0$. 

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(d) It follows from (1) and (2) that when $\alpha > 1 - c$,

$$f''(x) = x^{\alpha - 2 + c} \left( \alpha \cdot \frac{\sin(x^c)}{x^c} + c \cdot \cos(x^c) \right)$$

exists if and only if $\alpha \geq 2 - c$. When $\alpha = 1 - c$ and $c \geq 0$,

$$f''(x) = \lim_{x \to 0} \frac{1}{x} \left[ \alpha \cdot \left( \frac{\sin(x^c)}{x^c} - 1 \right) + c(\cos(x^c) - 1) \right]$$

$$= \lim_{x \to 0} x^{2c-1} \left( -\frac{\alpha}{6} - \frac{c}{2} \right)$$

exists if and only if $c \geq 1/2$ (note that $\frac{\alpha}{6} + \frac{c}{2} = \frac{1-c}{6} + \frac{c}{2} = \frac{1+2c}{6} \neq 0$).

2. Let $a \in \mathbb{R}$ be a given real number. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f'(x) < 0 < f''(x) \quad \text{if } x < a,$$

and

$$f'(x) > 0 > f''(x) \quad \text{if } x > a.$$

Prove that $f'(a)$ does not exists.

**Hint:** To see informally the reason why $f'(a)$ doesn’t exist, draw the graph of, for example,

$$f(x) = \begin{cases} 
(x - a)^2 & \text{if } x < a, \\
1 - \frac{1}{1+x-a} & \text{if } x \geq a.
\end{cases}$$

The monotonicity of $f(x)$ on $(a, \infty)$ and $(-\infty, a)$ dictates $f'(a) = 0$ if the derivative exists. But $f'(a) = 0$ means that $a$ is a local maximum of $f$, and hence $f''(0) < 0$. This is impossible since $f''$ is decreasing and positive on $x < a$. To make a formal proof from this argument, use the mean value theorem $\frac{f(x) - f(a)}{x-a} = f'(y_x)$ and the fact that the derivative is monotone on each side of $a$, to deduce that if $f'(a)$ exists then

$$f'(a) = \lim_{x \to a} f'(y_x) = \sup_{x < a} f'(x) = \inf_{x > a} f'(x) = 0.$$

Use then the mean value theorem for the derivative $\frac{f(x) - f'(a)}{x-a} = f''(z_x)$ to check that the assertion $f'(a) = 0$ contradicts the assumptions.

**Solution:** Since

$$0 < f''(x) \quad \text{if } x < a,$$

$f'(x)$ is increasing for $x < a$. Since

$$0 > f''(x) \quad \text{if } x > a,$$

$f'(x)$ is decreasing for $x > a$. Therefore, $f'(a)$ cannot exist.
$f'(x)$ is decreasing for $x > a$.

To get a contradiction, assume that $f'(a)$ exists and is equal to $b$. Then

$$b = \lim_{x \to \infty} \frac{f(x) - f(a)}{x - a}.$$ 

By the mean value theorem, for any $x \neq 0$, we have

$$\frac{f(x) - f(a)}{x - a} = f'(y_x),$$

for some $y_x$ within the interval containing $a$ and $x$ as its endpoints. Since $f'(x)$ is monotone and bounded on the both half-lines $x > a$ as well as $x < a$, this implies that

$$f'(a) = 0 = \sup_{x < a} f'(x) = \inf_{x > a} f'(x).$$

But then, the mean value theorem implies that for any $x > a$ there exists $z_x \in (a, x)$ such that

$$f''(z_x) = \frac{f'(x) - f'(a)}{x - a} = \frac{f'(x)}{x - a} > 0,$$

in contradiction to the assumption that $f''(z) < 0$ for any $z > a$. The calculation formally confirms the heuristic suggested by the picture (see the Hint) that the problem with the existence of the derivative is caused by the assumption that $f$ is concave for $x > a$.

3.

(a) Let $g(y) = \arcsin y^3$. Compute $g'(y)$ for $y \in (0, 1)$.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there exists $n \in \mathbb{N}$ such that

$$f(tx) = t^n f(x)$$

for any $t > 0$ and $x \in \mathbb{R}$. Show that $xf'(x) = nf(x)$ for all $x \in \mathbb{R}$.

Solution:

(a) $g(y) = \arcsin y^3$ and hence by the definition of the arcsin function, $g(y) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. In calculations below we occasionally use the fact that this implies $\cos g(y) \geq 0$.

First method: Use a table of derivatives or the inverse function theorem to verify that

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$ 

The chain rule therefore implies:

$$(\arcsin y^3)' = (y^3)' \cdot \frac{1}{\sqrt{1 - (y^3)^2}} = \frac{3y^2}{\sqrt{1 - y^6}}.$$
Second method: Observe that \( \sin g(y) = y^3 \). Differentiate this identity to obtain

\[ g'(y) \cos g(y) = 3y^2. \]

Conclude that

\[ g'(y) = \frac{3y^2}{\cos g(y)} = \frac{3y^2}{\sqrt{1 - \sin^2 g(y)}} = \frac{3y^2}{\sqrt{1 - y^6}}. \]

Third method: First, observe that \( \sin g(y) = y^3 \) and hence \( \sqrt[3]{\sin g(y)} = y \). This implies that

\[ g(y) = f^{-1}(y), \]

where \( f(x) = \sqrt[3]{\sin x} \). Next, by the chain rule,

\[ f'(x) = \frac{1}{3}(\sin x)^{-2/3} \cos x = \frac{1}{3}(\sin x)^{-2/3} \sqrt{1 - \sin^2 x}. \]

Thus, if \( f(x) = y = \sqrt[3]{\sin x} \), then

\[ f'(x) = \frac{1}{3}(\sin x)^{-2/3} \sqrt{1 - \sin^2 x} = \frac{1}{3} y^{-2} \sqrt{1 - y^6}. \]

It follows from the inverse function theorem, that

\[ g'(y) = (f^{-1})'(y) = \frac{1}{f'(x)} = \frac{3y^2}{\sqrt{1 - y^6}}. \]

(b) Differentiate both sides of the identity \( f(tx) = t^n f(x) \) with respect to \( t \). This yields:

\[ x f'(tx) = nt^{n-1} f'(x). \]

Now plug in \( t = 1 \) into the last identity.

4. Solve Exercise 80 in Chapter 2 of the textbook.

Solution:

(a) Suppose that the graph \( G_f = \{(p, y) \in M \times \mathbb{R} : y = fp\} \) is a union of two clopen subsets of \( M \times \mathbb{R} \). Say,

\[ H = \{(p, y) \in A \times \mathbb{R} : y = fp\} \]

and

\[ F = \{(p, y) \in B \times \mathbb{R} : y = fp\}, \]

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where $A$ and $B$ are two disjoint subsets of $M$ whose union is entire $M$. We will next show that if $f$ is continuous, then $A$ and $B$ are clopen in $M$, and hence $M$ is a disconnected set in contrary to the assumption. To this end, observe that if $p_n \in A$ and $\lim_{n \to \infty} p_n = p \in M$, then, since $f$ is continuous, $\lim_{n \to \infty} f(p_n) = f(p)$. Since $(p_n, f(p_n)) \in H$ and $H$ is a closed set, $(p, f(p)) \in H$. Hence $p \in A$. Thus $A$ is a closed set. Similarly, $B$ is closed. In particular, no point of $A$ is a cluster point of $B$. Thus all points of $A$ are its interior points, and hence $A$ is open.

(b) The result is actually true if $M \subset \mathbb{R}$. Therefore, to construct a counterexample, one has to use higher-dimensional domains. Consider, for instance, $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x, y) = \begin{cases} y & \text{if } x \geq 0 \text{ and } y \geq 0, \\
0 & \text{otherwise.} \end{cases}$$

The graph of this function is connected, but the function is discontinuous, for example, at $(0, 1)$. Indeed, $f(0, 1) = 1$ but $f(-\varepsilon, 1) = 0$ for any $\varepsilon > 0$. Notice that two parts of the graph (corresponding, respectively, to the arguments in the first quadrant and the rest of $\mathbb{R}^2$) meet at the origin, and hence the graph is connected.

We remark in passing that it is not hard to show that if $f : M \to \mathbb{R}$ is continuous and $G_f$ is connected, then $M$ must be connected.

(c) Suppose that $M$ is path-connected. Take any two points $p, q \in M$. Let $h : [a, b] \to M$ be a path connecting $p \in M$ and $q \in M$. That is, $h$ is continuous, $p = h(a)$, and $q = f(b)$. Define $g : [a, b] \to M$ by setting $g(x) = f(h(x))$ and $F : [a, b] \to M \times \mathbb{R}$ by setting

$$F(x) = (h(x), g(x)).$$

Then $F : [a, b] \to M \times \mathbb{R}$ is a path which connects $(p, f(p))$ and $(q, f(q))$ in the graph of the functions $f$. Since $(p, f(p))$ and $(q, f(q))$ are arbitrary points in the graph, the graph is path-connected.

(d) Recall that a function $F(x) = (F_1(x), F_2(x)) : [a, b] \to M \times \mathbb{R}$ is continuous if and only if both its components $F_1 : [a, b] \to M$ and $F_2 : [a, b] \to \mathbb{R}$ are continuous. Thus, if $F$ is a path joining $(p, f(p))$ and $(q, f(q))$ in the graph of a continuous function $f : M \to \mathbb{R}$, then $F_1 : [a, b] \to \mathbb{R}$ is a path joining $p$ and $q$ in $M$. Therefore, $M$ is a path-connected set if the graph of $f$ is path-connected. However, $f$ doesn’t have to be continuous if both $M$ and $G_f$ are path-connected. To show this, one may exploit the same counterexample as in part (b).

5. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose that

(i) $f$ is continuous for $x \geq 0$.

(ii) $f'(x)$ exists for $x > 0$.

(iii) $f(0) = 0.$
(iv) $f'$ is monotonically increasing.

Let

$$g(x) = \frac{f(x)}{x}, \quad x > 0.$$ 

Prove that $g$ is monotonically increasing.

**Solution:** We have

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}.$$ 

Hence, it suffices to show that

$$xf'(x) - f(x) \geq 0 \quad \text{for all } x > 0. \quad (3)$$

To this end, observe that by the mean value, for any $x \in \mathbb{R}$ there exists $y_x \in (0, x)$ such that

$$f'(y_x) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$ 

Hence

$$xf'(x) - f(x) = x(f'(x) - f'(y_x)),$$

and (3) follows from the monotonicity of the derivative (condition (iv) in the statement of the problem).

6. Let $X$ be the set of all bounded real-valued functions on a non-empty set $S$.

(a) For $x, y \in X$ set $d(x, y) = \sup_{t \in S} |x(t) - y(t)|$. Show that $d$ is a metric on $X$.

(b) For $x \in X$, let

$$f(x) = \inf_{t \in S} x(t) \quad \text{and} \quad g(x) = \sup_{t \in S} x(t).$$

Show that $f$ and $g$ are uniformly continuous functions from $(X, d)$ to $\mathbb{R}$.

**Solution:**

(a) Notice that, since functions in $X$ are bounded, $d(x, y) < +\infty$ for any $x, y \in X$. Furthermore, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x(t) = y(t)$ for all $t \in S$. Clearly, $d(x, y) = d(y, x)$. Finally, for any $t \in S$,

$$|x(t) - z(t)| \leq |x(t) - y(t)| + |y(t) - z(t)| \leq d(x, y) + d(y, z),$$

and hence

$$d(x, z) = \sup_{t \in S} |x(t) - z(t)| \leq d(x, y) + d(y, z).$$
(b) First, observe that for any \( x, y \in X \) we have

\[
\inf_{t \in S} x(t) = \inf_{t \in S} [(x(t) - y(t)) + y(t)] \geq \inf_{t \in S} [x(t) - y(t)] + \inf_{t \in S} y(t). \tag{4}
\]

Indeed, for any \( s \in S \),

\[
\inf_{t \in S} [x(t) - y(t)] + \inf_{t \in S} y(t) \leq [x(s) - y(s)] + y(s) = x(s),
\]

showing that \( \inf_{t \in S} [x(t) - y(t)] + \inf_{t \in S} y(t) \) is a lower bound for \( \{x(t) : t \in S\} \).

It follows from (4) that

\[
f(x) - f(y) = \inf_{t \in S} x(t) - \inf_{t \in S} y(t) \geq \inf_{t \in S} [x(t) - y(t)] = -\sup_{t \in S} [y(t) - x(t)] \geq -\sup_{t \in S} |y(t) - x(t)|.
\]

Similarly,

\[
f(y) - f(x) \geq -\sup_{t \in S} |y(t) - x(t)|.
\]

Combining these two inequalities, we obtain

\[
|f(y) - f(x)| \leq \sup_{t \in S} |y(t) - x(t)| = d(x, y).
\]

Thus, for any \( \varepsilon > 0 \), \( |f(y) - f(x)| \leq \varepsilon \) whenever \( d(x, y) \leq \varepsilon \).

Since \( g(x) = -f(-x) \), the claim about \( g \) follows directly from its counterpart for \( f \).