1. Let $\alpha$ and $c$ be real numbers, $c > 0$, and $f$ is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^\alpha \sin(x^c) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find all the values of the parameters $\alpha$ and $c$ such that:

(a) $f$ is continuous.

(b) $f'(0)$ exists.

(c) $f'$ is continuous.

(d) $f''(0)$ exists.

2. Let $a \in \mathbb{R}$ be a given real number. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f'(x) < 0 < f''(x) \quad \text{if } x < a,$$

and

$$f'(x) > 0 > f''(x) \quad \text{if } x > a.$$ 

Prove that $f'(a)$ does not exist.

*Hint: To see informally the reason why $f'(a)$ doesn’t exist, draw the graph of, for example,

$$f(x) = \begin{cases} (x-a)^2 & \text{if } x < a, \\ 1 - \frac{1}{1+x-a} & \text{if } x \geq a. \end{cases}$$

The monotonicity of $f(x)$ on $(a, \infty)$ and $(-\infty, a)$ dictates $f'(a) = 0$ if the derivative exists. But $f'(a) = 0$ means that $a$ is a local maximum of $f$, and hence $f''(0) < 0$. This is impossible since $f''$ is decreasing and positive on $x < a$. To make a formal proof from this argument, use the mean value theorem $\frac{f(x)-f(a)}{x-a} = f'(y_x)$ and the fact that the derivative is monotone on each side of $a$, to deduce that if $f'(a)$ exists then

$$f'(a) = \lim_{x \to a} f'(y_x) = \sup_{x < a} f'(x) = \inf_{x > a} f'(x) = 0.$$ 

Use then the mean value theorem for the derivative $\frac{f(x)-f(a)}{x-a} = f''(z_x)$ to check that the assertion $f'(a) = 0$ contradicts the assumptions.
(a) Let \( g(y) = \arcsin y^3 \). Compute \( g'(y) \) for \( y \in (0, 1) \).

(b) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and that there exists \( n \in \mathbb{N} \) such that

\[
f(tx) = t^n f(x)
\]

for any \( t > 0 \) and \( x \in \mathbb{R} \). Show that \( xf'(x) = nf(x) \) for all \( x \in \mathbb{R} \).

4. Solve Exercise 80 in Chapter 2 of the textbook.

5. Consider a function \( f : \mathbb{R} \to \mathbb{R} \). Suppose that

(i) \( f \) is continuous for \( x \geq 0 \).

(ii) \( f'(x) \) exists for \( x > 0 \).

(iii) \( f(0) = 0 \).

(iv) \( f' \) is monotonically increasing.

Let

\[
g(x) = \frac{f(x)}{x}, \quad x > 0.
\]

Prove that \( g \) is monotonically increasing.

6. Let \( X \) be the set of all bounded real-valued functions on a non-empty set \( S \).

(a) For \( x, y \in X \) set \( d(x, y) = \sup_{t \in S} |x(t) - y(t)| \). Show that \( d \) is a metric on \( X \).

(b) For \( x \in X \), let

\[
f(x) = \inf_{t \in S} x(t) \quad \text{and} \quad g(x) = \sup_{t \in S} x(t).
\]

Show that \( f \) and \( g \) are uniformly continuous functions from \((X, d)\) to \( \mathbb{R} \).