Homework #3

Solutions

EXERCISES FROM CHAPTER 2

10.

(a) Let \( X \) be any set that consist of at least two elements. Consider a discrete metric space \((X, d)\), where

\[
d(x, y) = \begin{cases} 
1 & \text{if } x \neq y, \\
0 & \text{otherwise}
\end{cases}
\]

It is easy to check that any subset of \((X, d)\) is clopen (Exercise 12 in Chapter 2 of the textbook). Therefore, by the result in Exercise 92(a) (Chapter 2 in the textbook), the boundary of any set in \((X, d)\) is empty. Notice that \( M_1(p) = X \setminus \{p\} \). Since \( X \) is assumed to include at least two elements, \( M_1(p) \) is not empty.

**Alternative example:** Consider \( \mathbb{R} \) with removed open intervals \((-2, -1) \) and \((1, 2)\), and use in this space the usual Euclidean metric inherited from \( \mathbb{R} \). The closure of the open ball \( B_2(0) = (-2, 2) \) in this space is \( B_1(0) = [-1, 1] \). Since \( B_1(0) \cap B_2(0)^c = \emptyset \), the boundary of \( B_2(0) \) is an empty set. In contrary, the sphere of radius 2 with center in origin is non-empty and consists of two points \( 2 \) and \(-2\).

(b) Yes. Indeed, the boundary is included in \( \overline{M_r(p)} = M_r(p) \) and in \( \overline{(M_r(p))^c} = (M_r(p))^c \).

In other word, if \( q \in \partial M_r(p) \) then \( q \) is a limit point of both

\[
M_r(p) = \{ s : d(d, s) < r \} \quad \text{and} \quad (M_r(p))^c = \{ s : d(d, s) \geq r \}.
\]

From \( q \in \lim M_r(p) \) it follows that \( d(p, q) \leq r \), and from \( q \in \lim (M_r(p))^c \) it follows that \( d(p, q) \geq r \). Thus \( d(p, q) = r \).

The following example shows that boundary can be an empty set, but that is a trivial case since an empty set is a subset of any set.

**Example 0.1.** Let \((X, d)\) be a metric space equipped with the discrete metric \( d \), as in part (a). Then each subset \( S \) is clopen, that is both \( S \) and \( S^c \) are closed, and hence

\[
\partial S = \overline{S} \cap \overline{S}^c = S \cap S^c = \emptyset.
\]

13. We use the usual Euclidean metric \( d(m, n) = |m - n| \). Let \( S \subset \mathbb{N} \) and \( n \in S \). Clearly \( M_1(n) = n \in S \). Thus \( S \) is an open set. It is also closed since \( S = (S^c)^c \) and \( S^c \) is open.
Any function \( f : \mathbb{N} \to \mathbb{N} \) is continuous because pre-image of any set is open.

14.

(a) First, assume that

\[
\text{dist}(p, S) = \inf \{d(p, s) : s \in S\} = 0.
\]

By the definition of the infimum, there is a sequence of point \( s_n \in S \) such that \( d(p, s_n) \leq 1/n \), and hence \( p = \lim_{n \to \infty} s_n \). Thus \( p \) is a limit point of \( S \).

Conversely, if \( p \) is a limit point of \( S \) then there exists a sequence of point \( s_n \in S \) such that \( p = \lim_{n \to \infty} s_n \). In particular, for any \( \delta > 0 \) and all \( n \in \mathbb{N} \) large enough we have \( d(p, s_n) \leq \delta \), and hence

\[
\text{dist}(p, S) = \inf \{d(p, s) : s \in S\} \leq \delta.
\]

Since \( \delta > 0 \) is arbitrary, this yields \( \text{dist}(p, S) = 0 \).

(b) The result follows from the following inequality:

\[
\left| \text{dist}(p, S) - \text{dist}(q, S) \right| \leq d(p, q). \tag{1}
\]

To prove (1), observe that for any \( s \in S \) we have

\[
d(p, s) \leq d(p, q) + d(q, s),
\]

and hence

\[
\inf_{s \in S} d(p, s) \leq \inf_{s \in S} [d(p, q) + d(q, s)] = d(p, q) + \inf_{s \in S} d(p, s).
\]

By definition of \( \text{dist}(\cdot, S) \) this means

\[
\text{dist}(p, S) \leq d(p, q) + \text{dist}(q, S),
\]

and hence

\[
\text{dist}(p, S) - \text{dist}(q, S) \leq d(p, q). \tag{2}
\]

Similarly, one can prove that

\[
\text{dist}(q, S) - \text{dist}(p, S) \leq d(p, q). \tag{3}
\]

Combining (2) and (3) together, we obtain (1).
(a) The condition $\lim_{n \to \infty} \text{diam } E_n = 0$ clearly implies that the intersection $\bigcap_n E_n$ consists of at most one point. The main difficulty is to show that the intersection is not empty. You have to assume of course that $E_n$ are non-empty. Let $x_1$ be any point in $E_1$. Set $P_1 = \{x_1\}$. If possible, choose $x_2 \in P_1^c \cap E_2$. If this is not possible then $E_2 = P_1$ and hence $E_n = P_1$ for all $n \in \mathbb{N}$, and we are done. If this is possible, set $P_2 = \{x_1, x_2\}$.

Now, assume that we have chosen a subset $P_n = \{x_1, \ldots, x_n\}$ of $E_n$ consisting of exactly $n$ distinct point ($n \geq 2$). If possible, choose any $x_{n+1} \in P_n^c \cap E_{n+1}$. If this is not possible then $E_{n+1} = P_n$ and hence $E_k = P_n$ for all $k \geq n$, and we are done since

$$\lim_{k \to \infty} \text{diam } E_k = \text{diam } P_n \neq 0,$$

which is impossible. If $x_{n+1} \in P_n^c \cap E_{n+1}$ exists, set $P_{n+1} = \{x_1, x_2, \ldots, x_{n+1}\}$. Since $P_n \subset E_n$ and $\lim_{k \to \infty} \text{diam } E_k = 0$, the sequence $\{x_n\}$ is Cauchy. Hence it converges to a point, say $x$.

Now, suppose that $x \notin E_k$ for some $k \in \mathbb{N}$. In a similar manner, as in the above solution of Problem 3(a) one can show that then $\inf_{y \in E_k} d(x, y) > 0$, for otherwise $x$ would be a limit point of $E_k$ and hence would belong to it since $E_k$ is closed. But then, since $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$ and $\lim_{k \to \infty} \text{diam } E_k = 0$, we would have

$$\lim_{n \to \infty} \inf_{y \in E_n} d(x, y) > 0.$$

This is however impossible since $x = \lim_{n \to \infty} x_n$ and $x_n \in E_n$ for all $n \in \mathbb{N}$. Hence

$$x \in \bigcap_n E_n.$$

The proof is complete.

(b) Since $f$ is a bijection, the inverse $f^{-1}$ is well-defined and is also a bijection. Consider now arbitrary $x, y \in \mathbb{N}$ and denote $p = f^{-1}x, q = f^{-1}y$. The isometry identity $d_N(fp, fq) = d_M(p, q)$ implies for $x = fp$ and $y = fq$,

$$d_N(x, y) = d_M(f^{-1}x, f^{-1}y).$$

Thus $f^{-1}$ is also an isometry. By the result in (a) both $f$ and $f^{-1}$ are continuous, and hence $f$ is a homeomorphism by definition.

(c) We have seen that if $f : [0, 1] \to [0, 2]$ is an isometry then $f^{-1}$ is also an isometry. This implies, in particular,

$$|f^{-1}(0) - f^{-1}(2)| = 2,$$

which is impossible since both $f^{-1}(0)$ and $f^{-1}(2)$ are points within the interval $[0, 1]$.

33. Recall that a metric space called complete if any Cauchy sequence converges in it. For instance, Euclidean spaces $\mathbb{R}^d, d \geq 1$, are complete. We will need the following lemma.
Lemma. Let $S$ be a subset of a metric space $M$ and $N$ be a complete metric space. Let $f : S \to N$ be a uniformly continuous function and let $(p_n)_{n \in \mathbb{N}}$ be any converging sequence of elements $p_n \in S$. Then $\lim_{n \to \infty} f(p_n)$ exists in $N$.

Proof of the lemma. Since $f$ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d_M(p, q) < \delta_\varepsilon \quad \text{implies} \quad d_N(fp, fq) < \varepsilon. \quad (4)$$

Let $p = \lim_{n \to \infty} \in M$. Then $\lim_{n \to \infty} d_M(p_n, p) = 0$. In particular, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$m, n > N_\varepsilon \quad \text{implies} \quad d_M(p_n, p_m) \leq d_M(p_n, p) + d_M(p, p_m) < \delta_\varepsilon.$$ 

Combining this with (4) yields

$$m, n > N_\varepsilon \quad \text{implies} \quad d_N(fp_n,fq_n) < \varepsilon.$$ 

Since $\varepsilon > 0$ is an arbitrary positive constant, it follows that $f(p_n)$ is a Cauchy sequence in $N$. Since $N$ is complete, the sequence converges to an element of $N$. \qed

I will prove (a) and (b) directly for a general complete metric space $N$ and $f : S \to N$, as it is suggested in part (c) of the exercise.

(a) By Proposition 12 (p. 67), $\overline{S} = \lim S$. For $p \in S$ put $\bar{f}(p) := f(p)$. To extend $f$ to $p \in \overline{S} \setminus S$, consider any sequence $p_n$ of elements of $S$ converging to $p$ and set

$$\bar{f}(p) := \lim_{n \to \infty} f(p_n). \quad (5)$$

The limit exists by the above lemma. Furthermore, the limit is independent of a particular choice of the sequence $p_n$. For if $(q_n)_{n \in \mathbb{N}}$ is another sequence of elements of $S$ converging to $p$, define a “merged” sequence $(r_n)_{n \in \mathbb{N}}$ as follows:

$$r_{2n-1} = p_n \quad \text{and} \quad r_{2n} = q_n, \quad n \in \mathbb{N}.$$ 

Then $\lim_{n \to \infty} r_n = p$. By the lemma above, $f(r_n)$ converges. Therefore, any subsequence of $f(r_n)$ converges to the same limit. Thus

$$\lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} f(q_n) = \lim_{n \to \infty} f(r_n).$$

It remains to check that $\bar{f}$ is uniformly continuous. To this end, take any $\varepsilon > 0$ and let $\delta_\varepsilon$ be as in the proof of the above lemma. Let $p, q \in \overline{S}$ such that $d_M(p, q) < \delta_\varepsilon$. Consider any sequences $p_n \in S$ and $q_n \in S$ such that

$$\lim_{n \to \infty} p_n = p \quad \text{and} \quad \lim_{n \to \infty} q_n = q. \quad (6)$$

If $p \in S$, simply put $p_n \equiv p$ for all $n \in \mathbb{N}$. Similarly, if $q \in S$, put $q_n \equiv q$ for all $n \in \mathbb{N}$. Then (5) and the independence of the limit of the choice of $p_n$ imply that

$$d_N(\bar{f}p, \bar{f}q) = \lim_{n \to \infty} d_N(fp_n,fq_n). \quad (7)$$

Furthermore, (6) and the fact that $d_M(p, q) < \delta_\varepsilon$ imply that for all $n$ large enough $d_M(p_n, q_n) < \delta_\varepsilon$, and hence $d_N(fp_n,fq_n) < \varepsilon/2$ by virtue of the definition of $\delta_\varepsilon$. It then follows from (7) that $d_N(\bar{f}p, \bar{f}q) \leq \varepsilon/2 < \varepsilon.$
Remark. To verify (7), use the triangle inequality. On one hand,
\[ d_N(\bar f p, \bar f q) \leq d_N(\bar f p, \bar f p_n) + d_N(\bar f p_n, \bar f q_n) + d_N(\bar f q_n, \bar f q), \]
and hence
\[ d_N(\bar f p, \bar f q) - d_N(\bar f p_n, \bar f q_n) \leq \left[ d_N(\bar f p, \bar f p_n) + d_N(\bar f q_n, \bar f q) \right] \to n \to \infty 0. \]
On the other hand,
\[ d_N(\bar f p_n, \bar f q_n) \leq d_N(\bar f p_n, \bar f p) + d_N(\bar f p, \bar f q) + d_N(\bar f q, \bar f q_n), \]
and hence
\[ d_N(\bar f p, \bar f q) - d_N(\bar f p_n, \bar f q_n) \geq -\left[ d_N(\bar f p_n, \bar f p) + d_N(\bar f q_n, \bar f q) \right] \to n \to \infty 0. \]

(b) This is easy. Any continuous extension \( \bar f : \overline{S} \to N \) of \( f \) should satisfy (5) by the “sequential” definition of the continuity. Thus \( \bar f \) is uniquely determined by (5).

91.

(a) By definition, \( \partial S \) consists of points in \( S \) that are are limit points of both \( S \) and \( S^c \). By Lemma 8 in the textbook (p. 62), \( \partial S \) is therefore the set of points of \( S \) such that any open neighborhood around these points has a non-empty intersection with both \( S \) and \( S^c \). Thus \( S \setminus \partial S \) is the set of points in \( S \) such that some open neighborhood of them lies entirely within \( S \). The latter characterization of \( S \setminus \partial S \) is however nothing but the definition of \( \text{int} \ S \).

(b) We must show that
\[ (\text{int} \ S)^c = \overline{S^c}. \]

By the definition of the interior set, \( (\text{int} \ S)^c \) is the collection of points in \( S \) such that any open neighborhood around them intersects \( S^c \). By Lemma 8 in the textbook (p. 62) this is exactly the closure of \( S^c \).

(c) The closure \( \overline{S^c} \) is a closed set by Proposition 12 (p. 67) and Theorem 7 (p. 62) in the textbook. Hence, the result in (b) implies that \( \text{int} \ S \) is an open set. However, if a set \( A \) is open then any ist point is an interior point by definition. Thus \( \text{int} \ A = A \). Applying this result to the open set \( A = \text{int} \ S \) completes the proof of claim (c).

(d) This is merely an observation that a point \( p \) is an interior point of \( S \cap T \) if and only if it is an interior point of both \( S \) and \( T \).

(e) Using the result in (b) we obtain the following dual claims for the closure:
Dual for (a):

$$
\overline{S^c} = (\text{int } S)^c = (S \setminus \partial S)^c = S^c \cup \partial S,
$$

which is equivalent to $$\overline{S} = S \cup \partial S.$$ Since by the definition of the boundary given on p. 66, $$\partial S^c = \partial S,$$ we finally obtain

$$\overline{S} = S \cup \partial S.$$

Dual for (c):

$$
\overline{S^c} = (\text{int } S)^c = (\text{int } (\text{int } S))^c = (\text{int } S)^c = S^c,
$$

which is equivalent to $$\overline{S} = S$$ or

$$\overline{S} = \overline{S}.$$

Dual for (d):

$$
(S \cap T)^c = (\text{int } S \cap T)^c = (\text{int } S \cap \text{int } T)^c = (\text{int } S)^c \cup (\text{int } T)^c
= \overline{S^c} \cup \overline{T^c}.
$$

Since $$(S \cap T)^c = S^c \cup T^c,$$ this is equivalent to $$\overline{S^c} \cup \overline{T^c} = \overline{S^c} \cup \overline{T^c}$$ or

$$\overline{S \cup T} = \overline{S} \cup \overline{T}.$$

(f) Clearly, if $$p$$ is an interior point of either $$S$$ or $$T$$ it is an interior point of their union. This yields the inclusion $$\text{int } (S \cup T) \supset (\text{int } S \cup \text{int } T).$$ To see that the inclusion can be strict, consider, for instance, the following two subsets of the real line: $$S = [1, 2)$$ and $$T = (0, 1]$$. Then

$$\text{int } S \cup \text{int } T = (0, 1) \cup (1, 2) = (0, 2) \setminus \{1\},$$

while $$\text{int } (S \cup T) = (0, 2).$$