This is a take-home examination. The exam includes 8 questions. The total mark is 100 points. Please show all the work, not only the answers.

1. [12 points] Fix any \( d \in \mathbb{N} \) and let \( \| \cdot \| \) denote the usual Euclidean norm in \( \mathbb{R}^d \). That is,
\[
\| x \| = \sqrt{\sum_{i=1}^{d} x_i^2}
\]
for a vector \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

Suppose that \( u, v \in \mathbb{R}^d \). Find \( w \in \mathbb{R}^d \) and \( r > 0 \) such that
\[
\| x - u \| = 2 \| x - v \|
\]
if and only if \( \| x - w \| = r \).

Hint: Prove that \( w = \frac{1}{3}(4v - u) \) and \( r = \frac{2}{3} \| v - u \| \).

First solution: This is a short solution based on a “completion to square” idea.

Start with the equation
\[
\| x - u \|^2 = 4 \| x - v \|^2. \tag{1}
\]
Observe that it is equivalent to
\[
\| x \|^2 - 2x \cdot u + \| u \|^2 = 4\| x \|^2 - 8x \cdot v + 4\| v \|^2,
\]
and hence to
\[
\| x \|^2 - 2x \cdot \frac{4v - u}{3} = \frac{\| u \|^2 - 4\| v \|^2}{3}.
\]
Let \( w = \frac{1}{3}(4v - u) \). Then the left-hand side is \( \| x \|^2 - 2x \cdot w \). Add to both the sides \( \| w \|^2 \), to obtain
\[
\| x - w \|^2 = \| x \|^2 - 2x \cdot w + \| w \|^2 = \frac{\| u \|^2 - 4\| v \|^2}{3} + \| w \|^2
= \frac{\| u \|^2 - 4\| v \|^2}{3} + \frac{1}{9}\| 4v - u \|^2
= \frac{4}{9}\| v - u \|^2 = r^2,
\]
where \( r = \frac{2}{3} \| v - u \| \).
**Remark.** This solution is using the hint, in particular it enjoys the a-priori knowledge of \( w \). To guess the value of \( w \) and \( r \) one can find two points satisfying equation (1) on the straight line connecting \( u \) and \( v \) and then observe that \( w \) must be the center of the segment connecting these two points. The location of the two points is given by Equation (8) below. An easy way to find them is to set \( x = v + t(u - v) = u + (1 - t)(v - u) \), plug these expressions into, correspondingly, the right- and left-hand sides of equation (1), and solve the resulting quadratic equation to find two possible values of \( t \).

**Second solution:** This solution is longer than the previous one. It exploits the same orthogonal decomposition \( x = v - t(v - u) + z_\perp \) as the solution of a similar problem in the sample, and in fact is mimicking the latter.

Assume that
\[
\|x - u\|^2 = 4\|x - v\|^2. \tag{2}
\]

Write
\[
x = v - t(v - u) + z_\perp, \tag{3}
\]
where
\[
(v - u) \cdot z_\perp = 0
\]
and \( t \in \mathbb{R} \) is a suitable constant. Thus \(-t(v - u)\) and \( z_\perp \) are projection of the vector \( x - v \) into, respectively, the subspace spanned by \( v - u \) and its orthogonal complement.

Notice that (3) is equivalent to
\[
x = u + (1 - t)(v - u) + z_\perp. \tag{4}
\]

Thus, plugging the above expressions for \( x \) into (2) (respectively, (3) into the left-hand side and (4) into the right-hand side) and using the Pythagoras theorem, we obtain
\[
(1 - t)^2\|v - u\|^2 + \|z_\perp\|^2 = 4t^2\|v - u\|^2 + 4\|z_\perp\|^2.
\]

It follows that (2) is equivalent to
\[
\|z_\perp\|^2 = \frac{(1 + t)(1 - 3t)}{3}\|v - u\|^2. \tag{5}
\]

For a fixed value of the parameter \( t \), the right-hand side of (5) is a constant independent of \( x \). In general, in view of (3), for any constant \( a > 0 \),
\[
\|z_\perp\| = a
\]
is an equation of a sphere with center at \( v - t(v - u) \) in the \((d - 1)\)-dimensional subspace orthogonal to the vector \( v - u \).

Since the right-hand side expression in the equation (5) must be non-negative, we obtain that \((1 + t)(1 - 3t) \geq 0\), and hence
\[
-1 \leq t \leq 1/3. \tag{7}
\]
Notice that $\|\mathbf{x} - \mathbf{w}\| = r$ is an equation of the $d$-dimensional sphere with center at $\mathbf{w}$ and radius equal $r$. In particular, the values of $\mathbf{w}$ and $r$, if exist, are uniquely determined by (2). It follows from (6) and (7) that if such a sphere exists its center must correspond to the middle of the segment with the endpoints (corresponding to the endpoints of the interval in (7) describing the range of $t$):

$$x_1 = v - (-1)(v - u) \quad \text{and} \quad x_2 = v - \frac{1}{3}(v - u).$$  

(8)

By the symmetry, the center $\mathbf{w}$ of the sphere $\|\mathbf{x} - \mathbf{w}\| = r$ should be located in the mid-point of this segment, namely at $v - t_0(v - u)$ with $t_0 = \frac{-1+1/3}{2} = -\frac{1}{3}$ being the mid-point of the interval $(-1, 1/3)$. Thus

$$w = v - \left(-\frac{1}{3}\right) \cdot (v - u) = \frac{1}{3}(4v - u).$$

It follows then from (5) that

$$r^2 = \|\mathbf{x} - \mathbf{w}\|^2 = \left(t + \frac{1}{3}\right)^2 \|\mathbf{u} - \mathbf{v}\|^2 + \|\mathbf{z}_\perp\|^2 = \left(t + \frac{1}{3}\right)^2 \|\mathbf{u} - \mathbf{v}\|^2 + \frac{(1 + t)(1 - 3t)}{3} \|\mathbf{v} - \mathbf{u}\|^2 = \frac{4}{9} \|\mathbf{v} - \mathbf{u}\|^2,$$

and hence $r = \frac{2}{3}\|\mathbf{v} - \mathbf{u}\|$. As we mentioned before, the values of $\mathbf{w}$ and $r$ are uniquely determined by (2), and hence the problem is solved once one admissible pair $(\mathbf{w}, r)$ is found.

2. [14 points] Solve Exercise 35 in Chapter 1 of the textbook.

Remark. Solutions of Exercises 2 and 3 are somewhat similar, both are based on a version of Cantor’s diagonal trick. However, the solution to the former is slightly more technically involved, while the solution to the latter relies on the main idea in a more direct and transparent way. It therefore might be of a benefit to students to first read the solution to Exercise 3.

Solution:

(a) Functions with the values in the set $\{0, 1\}$ are commonly called binary functions. If $f$ is a binary function and $A = \{s \in S : f(s) = 1\}$, then $f$ is called the indicator of set $A$ and is denoted by $1_A$. Let $G: \mathcal{P} \rightarrow \mathcal{F}$ be a function mapping the subsets of $S$ into their indicators:

$$G(A) = 1_A.$$}

Notice that any binary function is an indicator of some (possibly empty) set. Furthermore, $1_A = 1_B$ clearly implies $A = B$. Thus $G$ is a bijection from $\mathcal{P}$ onto $\mathcal{F}$. The inverse function $G^{-1}: \mathcal{F} \rightarrow \mathcal{P}$ defined by $G^{-1}(1_A) = A$ is the desired bijection from $\mathcal{F}$ onto $\mathcal{P}$. The existence of the bijection shows that $\mathcal{F}$ and $\mathcal{P}$ are equivalent (have the same cardinality).
(b) First, notice that if $S$ is empty or finite then
\[ \text{card}(\mathcal{P}) = \text{card}(\mathcal{P}) = 2^{\text{card}(S)}. \]

The first equality is the result in (a) while the second can be easily verified, for instance, by induction on the number of elements in $S$.

In general, there is a trivial injection from $S$ onto $\mathcal{P}$ defined by $\gamma(s) = \{s\}$, thus mapping $s \in S$ into the singleton $\{s\} \in \mathcal{P}$. To show that $S$ is not equivalent to $\mathcal{F}$ (and hence is not equivalent to $\mathcal{P}$) one can utilize a version of Cantor’s diagonal trick as follows.

Assume that there is a bijection $\beta : S \to \mathcal{F}$. It will be convenient to denote $\beta(s)$ as $f(s)$. Then $f(s) : S \to \{0,1\}$ is a binary function for each $s \in S$, and, since $\beta$ is a bijection, each element in $\mathcal{F}$ is $f(s)$ for some $s \in S$. To see that this is in fact impossible, define a binary function $h : S \to \{0,1\}$ by setting
\[ h(s) = \text{not } f(s)(s) = 1 - f(s)(s). \] (9)

Since $\beta$ is a bijection, we must have $h = f(t)$ for some $t \in S$. However, if $h = f(t)$ then $h(t) = f(t)(t)$ contradicting (9). This shows that $h \neq f(t) = \beta(t)$ for any $t \in S$, and hence $\beta : S \to \mathcal{F}$ is actually not a bijection.

Remark. Here is an instructive alternative solution to 2(b). It is based on a similar but different idea from the one hinted in the textbook. In particular, rather than compare $S$ to $\mathcal{F}$ it directly compares $S$ to $\mathcal{P}$.

Assume that there is a bijection $\alpha : S \to \mathcal{P}$. Define $E \in \mathcal{P}$ as follows
\[ E = \{s \in S : s \notin \alpha(s)\}. \] (10)

Since $\alpha$ is a surjection, $E = \alpha(s)$ for some $s \in S$. However,

- $s \in E = \alpha(s)$ directly contradicts (10).
- $s \notin \alpha(s)$ implies, by virtue of (10), that $s \in E = \alpha(s)$.

Either way we get a contradiction showing that, in fact, a bijection $\alpha : S \to \mathcal{P}$ doesn’t exist.

3. [12 points] Let $E$ be the set of all real numbers $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is $E$ countable?

Solution: One can utilize a version of Cantor’s diagonal trick as follows. Assume that $E$ is countable and let $f : \mathbb{N} \to E$ be a bijection from $\mathbb{N}$ to $E$ (enumeration of the elements in $E$). Using decimal expansions write
\[ f(n) = 0.a_{n,1}a_{n,2}a_{n,3} \cdots = \sum_{k=1}^{\infty} 10^{-k}a_{n,k}, \quad n \in \mathbb{N}, \]
where $a_{n,k} \in \{4, 7\}$, $k \in \mathbb{N}$. Consider the element of $E$

$$g = 0. b_1 b_2 b_3 \cdots = \sum_{k=1}^{\infty} 10^{-k} b_k,$$

where $b_k \in \{4, 7\}$ are defined as follows:

$$b_k = \text{not } a_{k,k} = 11 - a_{k,k}, \quad k \in \mathbb{N}. \quad (11)$$

Since $f$ is a bijection, we must have $g = f(n)$ for some $n \in \mathbb{N}$. However, if $g = f(n)$ then $b_n = a_{n,n}$ contradicting (11). This shows that $g \neq f(n)$ for any $n \in \mathbb{N}$, and hence $f$ is actually not a bijection.

4. [14 points] Fix any real number $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$ and define recursively

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = \frac{\alpha - x_n^2}{1 + x_n}, \quad n \in \mathbb{N}.$$

(a) Prove that $x_1 > x_3 > x_5 > \ldots$.

(b) Prove that $x_2 < x_4 < x_6 < \ldots$.

(c) Prove that $\lim_{n \to \infty} x_n = \sqrt{\alpha}$.

(d) Let $\varepsilon_n = |x_n - \sqrt{\alpha}|$. Show that $\varepsilon_n < c \beta^n$ for some constants $c > 0$ and $\beta \in (0, 1)$.

Solution:

(a, b) We have:

$$x_{n+1} - \sqrt{\alpha} = \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} = \frac{(\sqrt{\alpha} - 1)(\sqrt{\alpha} - x_n)}{1 + x_n} \quad (12)$$

Hence, since $\alpha > 1$,

The sequence $y_n = x_n - \sqrt{\alpha}$ alternates the signs \quad (13)

Furthermore,

$$x_{n+2} = \frac{\alpha + x_{n+1}}{1 + x_{n+1}} = \frac{\alpha + \frac{\alpha + x_n}{1 + x_n}}{1 + \frac{\alpha + x_n}{1 + x_n}} = \frac{2\alpha + x_n(1 + \alpha)}{2x_n + 1 + \alpha}, \quad (14)$$

and hence

$$x_{n+2} - x_n = \frac{2\alpha + x_n(1 + \alpha)}{2x_n + 1 + \alpha} - x_n = \frac{2(\alpha - x_n^2)}{2x_n + 1 + \alpha}. \quad (15)$$

The claims in (a) and (b) follow from (13) and (15) combined together.
(c) It follows from the results in (a) and (b) that the sequences $x_{2n}$ and $x_{2n+1}$ are monotone. Furthermore, (13) implies that

$$x_{2n} < \sqrt{\alpha} < x_{2n-1}, \quad \forall \ n \in \mathbb{N}.$$  

Therefore the sequence are bounded and following limits exist:

$$t_1 = \lim_{n \to \infty} x_{2n} \quad \text{and} \quad t_2 = \lim_{n \to \infty} x_{2n+1}.$$  

Moreover, it follows from (14) that $t_1$ and $t_2$ are non-negative roots of the equation

$$t = \frac{2\alpha + t(1 + \alpha)}{2t + 1 + \alpha}.$$  

The last identity yields $2t^2 + t(1 + \alpha) = 2\alpha + t(1 + \alpha)$, which has a unique non-negative solution $t = \sqrt{\alpha}$. The proof of part (c) is complete.

(d) Let $\varepsilon_n = |x_n - \sqrt{\alpha}|$. It follows from (12) that

$$\varepsilon_{n+1} = \frac{(\sqrt{\alpha} - 1)\varepsilon_n}{1 + x_n}.$$  

Therefore,

$$\lim_{n \to \infty} \frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{\sqrt{\alpha} - 1}{\sqrt{\alpha} + 1}, \quad (16)$$  

which implies that the error $\varepsilon_n$ decays asymptotically with an exponential rate. More precisely, pick any

$$\beta \in \left(\frac{\sqrt{\alpha} - 1}{\sqrt{\alpha} + 1}, 1\right).$$  

It follows from (16) and the definition of the limit that there is $N \in \mathbb{N}$ such that $n > N$ implies

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} < \beta < 1.$$  

Thus, for any $m \in \mathbb{N},$

$$\frac{\varepsilon_{N+m}}{\varepsilon_N} = \prod_{k=N}^{N+m-1} \frac{\varepsilon_{k+1}}{\varepsilon_k} < \beta^m.$$  

Using the substitution $n = N + m$ we can rewrite this as

$$\varepsilon_n < \frac{\varepsilon_N}{\beta^N} \beta^n, \quad n > N.$$  

Therefore, setting

$$c = \max_{1 \leq k \leq N} \left\{ \frac{\varepsilon_k}{\beta^k} \right\},$$  

we obtain

$$\varepsilon_n \leq c\beta^n, \quad \forall \ n \in \mathbb{N}.$$  

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5. [12 points] Fix any \( d \in \mathbb{N} \) and let \( \| \cdot \| \) denote the usual Euclidean norm in \( \mathbb{R}^d \). That is,
\[
\| x \| = \sqrt{\sum_{i=1}^{d} x_i^2}
\]
for a vector \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

We say that a sequence \((x_n)_{n \in \mathbb{N}}\) in \( \mathbb{R}^d \) converges to \( x \in \mathbb{R}^d \) and write \( \lim_{n \to \infty} x_n = x \) if for any \( \varepsilon > 0 \) there exists \( N_\varepsilon \in \mathbb{N} \) such that
\[
n \in \mathbb{N} \quad \text{and} \quad m, n > N_\varepsilon \implies \| x_n - x \| < \varepsilon.
\]

Call two converging sequences \( X = (x_n)_{n \in \mathbb{N}} \) and \( Y = (y_n)_{n \in \mathbb{N}} \) in \( \mathbb{R}^n \) equivalent and write \( X \sim Y \) if
\[
\lim_{n \to \infty} \| x_n - y_n \| = 0.
\]

Let \( E \) be the set of all converging sequences in \( \mathbb{R}^n \). Show that \( \sim \) is an equivalence relation in \( E \).

**Solution:** To show that \( \sim \) is an equivalence relation we must verify the following three property of the relation:

1. \( X \sim X \) for any \( X \in E \) (reflexivity)
2. \( X \sim Y \) implies \( Y \sim X \) for any \( X, Y \in E \) (symmetry)
3. \( X \sim Y \) and \( Y \sim Z \) imply together \( X \sim Z \) for any \( X, Y, Z \in E \) (transitivity)

Take any \( X = (x_n)_{n \in \mathbb{N}} \in E \). Clearly, \( \lim_{n \to \infty} \| x_n - x_n \| = 0 \) and hence \( X \sim X \). Furthermore, if \( Y = (y_n)_{n \in \mathbb{N}} \in E \) and \( \lim_{n \to \infty} \| x_n - y_n \| = 0 \) then \( \lim_{n \to \infty} \| y_n - x_n \| = 0 \), and hence \( X \sim Y \) implies \( Y \sim X \). Finally, if \( Z = (z_n)_{n \in \mathbb{N}} \in E \), the triangle inequality yields
\[
\| x_n - z_n \| \leq \| x_n - y_n \| + \| y_n - z_n \|,
\]
and hence \( X \sim Y \) and \( Y \sim Z \) imply together \( X \sim Z \).

6. [12 points]

(a) Let \( (s_n)_{n \in \mathbb{N}} \) be a sequence of reals such that
\[
s_{n+1} = \frac{s_n + s_{n-1}}{2}.
\]

Show that \( s_n \) is a Cauchy sequence and hence converges.

(b) Let \( (s_n)_{n \in \mathbb{N}} \) be a sequence of reals defined recursively by
\[
s_1 = 0, \quad s_{2n} = \frac{s_{2n-1}}{2}, \quad s_{2n+1} = \frac{1}{2} + s_{2n}.
\]

Find \( \limsup_{n \to \infty} s_n \) and \( \liminf_{n \to \infty} s_n \).

*Hint:* Consider the sequences \( u_n = s_{2n} \) and \( v_n = s_{2n-1} \) separately.
Solution:

(a) Let $I_n$ be the closed interval with endpoints $s_{n-1}$ and $s_n$. If $s_{n-1} = s_n$ the interval is degenerate and consists of a single point. Let $L_n$ be the length of the interval $I_n$, that is $L_n = |s_n - s_{n-1}|$. Then

$$L_{n+1} = \left| \frac{s_n + s_{n-1}}{2} - s_n \right| = \frac{1}{2} L_n.$$ 

Iterating, we obtain $L_{n+1} = 2^{-n} L_1$, and hence

$$\lim_{n \to \infty} L_n = 0. \quad (17)$$

Observe now that $s_{n+1}$ is the mid-point of the interval $I_n$, and hence $I_{n+1} \subset I_n$. In particular, $s_k \in I_n$ for any $n \in \mathbb{N}$ and $k > n$. By virtue of (17), this implies that $s_n$ is a Cauchy sequence and hence converges.

(b) Let $u_n = s_{2n}$ and $v_n = s_{2n-1}$. In this notation, the recursions in the statement of the problem can be rewritten as

$$u_n = \frac{v_n}{2} \quad \text{and} \quad v_{n+1} = \frac{1}{2} + u_n.$$ 

It follows that

$$u_{n+1} = \frac{v_{n+1}}{2} = \frac{1}{4} + \frac{1}{2} u_n \quad (18)$$

and

$$v_{n+1} = \frac{1}{2} + u_n = \frac{1}{2} + \frac{1}{2} v_n. \quad (19)$$

To understand the structure of the sequences $u_n$ and $v_n$ pick any constants $\alpha, \beta \in \mathbb{R}$ and set $u_n = \alpha + x_n$, $v_n = \beta + y_n$. In the new notation, (18) and (19) become, respectively,

$$\alpha + x_{n+1} = \frac{1}{4} + \frac{1}{2} \alpha + \frac{1}{2} x_n$$

and

$$\beta + y_{n+1} = \frac{1}{2} + \frac{1}{2} \beta + \frac{1}{2} y_n.$$ 

Thus, if we could choose parameters $\alpha$ and $\beta$ in such a way that

$$\alpha = \frac{1}{4} + \frac{1}{2} \alpha \quad (20)$$

and

$$\beta = \frac{1}{2} + \frac{1}{2} \beta, \quad (21)$$

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we would have \( x_{n+1} = \frac{1}{2}x_n \) and \( y_{n+1} = \frac{1}{2}y_n \). The latter identities clearly imply
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0. \tag{22}
\]
Fortunately, (20) and (21) both have solutions, namely \( \alpha = \frac{1}{2} \) and \( \beta = 1 \). Taking into account (22), this yields
\[
\lim_{n \to \infty} u_n = \alpha = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} v_n = \beta = 1. \tag{23}
\]
It is not hard to verify, using the definition of the limit of a sequence and (23), that any converging subsequence of \( s_n \) cannot include infinitely many elements from the sequence \( u_n \) and, at the same time, infinitely many elements from the sequence \( v_n \). This implies that any converging subsequence of \( s_n \) converges to either 1 or \( \frac{1}{2} \). Using the sequential definition of the lim sup and lim inf, this immediately yields
\[
\limsup_{n \to \infty} s_n = 1 \quad \text{and} \quad \liminf_{n \to \infty} s_n = \frac{1}{2}.
\]

7. [12 points] Let \( (s_n)_{n \in \mathbb{N}} \) be a sequence of reals and define
\[
t_n = \frac{1}{n} \sum_{i=1}^{n} s_i.
\]
(a) Prove that if \( \lim_{n \to \infty} s_n = s \) then \( \lim_{n \to \infty} t_n = s \).

(b) Give an example to show that \( t_n \) can converge even though \( s_n \) doesn’t.

Solution:

(a) Assume that \( \lim_{n \to \infty} s_n = s \). For \( \varepsilon > 0 \) and let \( N_\varepsilon \in \mathbb{N} \) be an integer such that \( n > N_\varepsilon \) implies \( |s_n - s| < \varepsilon \). Then for \( n > N_\varepsilon \), we have
\[
|t_n - s| = \left| \frac{1}{n} \sum_{i=1}^{n} (s_i - s) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |s_i - s| \leq \frac{1}{n} \sum_{i=1}^{N_\varepsilon} |s_i - s| + \frac{1}{n} \sum_{i=N_\varepsilon+1}^{n} |s_i - s| + \varepsilon.
\]
Taking \( n \to \infty \) while \( \varepsilon \) and \( N_\varepsilon \) remain fixed in the both sides of the resulting inequality
\[
|t_n - s| \leq \frac{1}{n} \sum_{i=1}^{N_\varepsilon} |s_i - s| + \varepsilon,
\]
we obtain
\[
\limsup_{n \to \infty} |t_n - s| \leq \varepsilon.
\]
Since \( \varepsilon > 0 \) is an arbitrary positive real, \( \limsup_{n \to \infty} |t_n - s| = 0 \) and hence
\[
\lim_{n \to \infty} |t_n - s| = 0.
\]
The latter result is equivalent to the desired assertion \( \lim_{n \to \infty} t_n = s \).

(b) Consider \( s_n = (-1)^n \). Clearly, the sequence \( s_n \) diverges. However, \( t_n = (-1)^{n+1}/n \) and hence \( \lim_{n \to \infty} t_n = 0 \).

8. [12 points]

(a) Use induction to show that if \( (x + 1/x) \) is integer then \( (x^n + 1/x^n) \) is also integer for any \( n \in \mathbb{N} \).

(b) Show that
\[
\sup(A \cup B) = \max\{\sup A, \sup B\}
\]
for any sets \( A, B \subset \mathbb{R} \).

**Solution:**

(a) Let \( S_n = (x^n + 1/x^n), n \in \mathbb{N} \). Verify that \( S_n \cdot S_1 = S_{n+1} + S_{n-1} \), and hence
\[
S_{n+1} = S_n \cdot S_1 - S_{n-1}.
\]
The claim follows from this identity by a version of the induction argument.

(b) The inequalities \( \sup A \leq \sup(A \cup B) \) and \( \sup B \leq \sup(A \cup B) \) follow from the inclusions \( A \subset (A \cup B) \) and \( B \subset (A \cup B) \). These two inequalities together imply that
\[
\max\{\sup A, \sup B\} \leq \sup(A \cup B). \quad (24)
\]
On the other hand, \( \sup A \) bounds from above any element of \( A \) while \( \sup B \) bounds from above any element of \( B \). It follows that \( \max\{\sup A, \sup B\} \) bounds from above any element in \( A \cup B \). Hence
\[
\sup(A \cup B) = \max\{\sup A, \sup B\}. \quad (25)
\]
The desired result follows from (24) and (25) combined together.