1. [12 points] Fix any $d \in \mathbb{N}$ and let $\| \cdot \|$ denote the usual Euclidean norm in $\mathbb{R}^d$. That is,
$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2} \text{ for a vector } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$  
Suppose that $u, v \in \mathbb{R}^d$. Find $w \in \mathbb{R}^d$ and $r > 0$ such that
$$\|x - u\| = 2\|x - v\|$$
if and only if $\|x - w\| = r$.

*Hint:* Prove that $w = \frac{1}{3}(4v - u)$ and $r = \frac{2}{3}\|v - u\|$.

2. [14 points] Solve Exercise 35 in Chapter 1 of the textbook.

3. [12 points] Let $E$ be the set of all real numbers $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is $E$ countable?

4. [14 points] Fix any real number $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$ and define recursively
$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}, \quad n \in \mathbb{N}.$$

(a) Prove that $x_1 > x_3 > x_5 > \ldots$.

(b) Prove that $x_2 < x_4 < x_6 < \ldots$.

(c) Prove that $\lim_{n \to \infty} x_n = \sqrt{\alpha}$.

(d) Let $\varepsilon_n = |x_n - \sqrt{\alpha}|$. Show that $\varepsilon_n < c\beta^n$ for some constants $c > 0$ and $\beta \in (0, 1)$.

5. [12 points] Fix any $d \in \mathbb{N}$ and let $\| \cdot \|$ denote the usual Euclidean norm in $\mathbb{R}^d$. That is,
$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2} \text{ for a vector } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$
We say that a sequence \((x_n)_{n \in \mathbb{N}}\) in \(\mathbb{R}^d\) converges to \(x \in \mathbb{R}^d\) and write \(\lim_{n \to \infty} x_n = x\) if for any \(\varepsilon > 0\) there exists \(N_\varepsilon \in \mathbb{N}\) such that
\[
n \in \mathbb{N} \quad \text{and} \quad m, n > N_\varepsilon \quad \Rightarrow \quad ||x_n - x|| < \varepsilon.
\]
Call two converging sequences \(X = (x_n)_{n \in \mathbb{N}}\) and \(Y = (y_n)_{n \in \mathbb{N}}\) in \(\mathbb{R}^n\) equivalent and write \(X \sim Y\) if
\[
\lim_{n \to \infty} ||x_n - y_n|| = 0.
\]

Let \(E\) be the set of all converging sequences in \(\mathbb{R}^n\). Show that \(\sim\) is an equivalence relation in \(E\).

6. [12 points]
(a) Let \((s_n)_{n \in \mathbb{N}}\) be a sequence of reals such that
\[
s_{n+1} = \frac{s_n + s_{n-1}}{2}.
\]
Show that \(s_n\) is a Cauchy sequence and hence converges.

(b) Let \((s_n)_{n \in \mathbb{N}}\) be a sequence of reals defined recursively by
\[
s_1 = 0, \quad s_2 = \frac{s_2 - 1}{2}, \quad s_{2n} = \frac{1}{2} + s_{2n-1}.
\]
Find \(\lim \sup_{n \to \infty} s_n\) and \(\lim \inf_{n \to \infty} s_n\).

Hint: Consider the sequences \(u_n = s_{2n}\) and \(v_n = s_{2n-1}\) separately.

7. [12 points] Let \((s_n)_{n \in \mathbb{N}}\) be a sequence of reals and define
\[
t_n = \frac{1}{n} \sum_{i=1}^{n} s_i.
\]
(a) Prove that if \(\lim_{n \to \infty} s_n = s\) then \(\lim_{n \to \infty} t_n = s\).

(b) Give an example to show that \(t_n\) can converge even though \(s_n\) doesn’t.

8. [12 points]
(a) Use induction to show that if \((x + 1/x)\) is integer then \((x^n + 1/x^n)\) is also integer for any \(n \in \mathbb{N}\).

(b) Show that
\[
\sup(A \cup B) = \max\{\sup A, \sup B\}
\]
for any sets \(A, B \subset \mathbb{R}\).