Math 265 Practice Exam 2

Answer each question completely. Show all work. No credit for mere answers with no work shown. Show the steps of calculations, simplify end results as far as reasonably possible. Give exact results like $\sqrt{5}$ or $\pi$, no rounded decimal fractions unless required, and then only round in the last step.

Problem 1 Consider the function

$$f(x, y) = y^2 e^{x^2-4y}.$$  

a) Find the gradient $\nabla f(x, y)$.

b) Find an equation for the tangent plane to the surface given by $z = f(x, y)$ in $(x, y) = (2, 3)$.

Problem 2 Consider the ellipsoids with equations $F(x, y, z) = 8$ and $G(x, y, z) = 8$, where

$$F(x, y, z) = x^2 + 4y^2 + (z - 2)^2$$
$$G(x, y, z) = (x + 2)^2 + 4y^2 + z^2$$

The point $P_0 = (-2, 1, 2)$ lies on the intersection of the two ellipsoids, which is an ellipse. Find a vector equation for the tangent line to this ellipse in the point $P_0$. 

Ellipsoids in blue, intersection in green, $P_0$ in red
Problem 3 A gas particle in the plane is experiencing a pressure $P = x^4 - x^2y^2 + y^4$ in position $(x, y)$. The particle is always moving in the direction of lowest pressure - assume the particle moves in the direction where the pressure decreases most rapidly.

a) Find a unit vector in the direction of the particle’s movement when it is at point $(2, 1)$.
b) Find all points $(x, y)$ where the particle would move parallel to the $x$-axis and sketch them (please include labeled axes with units).

Problem 4 The quantities $x, y$ are known to satisfy the equation $F(x, y) = 3$ with

$$F(x, y) = 3x + \cos(\pi(x + y)).$$

In a neighborhood of $x = 1, y = -1/2$, this determines $y = y(x)$ as a function of $x$ (so $y(1) = -1/2$).

a) Using partial derivatives of $F$, find $y' = dy/dx$ at $(x, y) = (1, -1/2)$.
b) Find an approximation to $y(1.02)$ using differentials $dy$ and $dx = 0.02$. Round to four digits after the decimal point.
c) Now let $x, y$ again be independent variables. Find an approximation to $F(2.04, -0.49)$ using differentials $dx$ and $dy$ at the point $p_0 = (2, -0.5)$. Round to four digits after the decimal point.

Problem 5 Consider the surface $S$ given by $F(x, y, z) = 9$ with

$$F(x, y, z) = x^2 + 2xz + y^2 + 2yz + 3z^2$$

a) Show that for $(x, y, z)$ on the surface $S$, the gradient of $F$ is never the zero vector.
b) Find all points $(x, y, z)$ on $S$ where the tangent plane is horizontal.

Problem 6 Consider the function $f(x, y) = x^2y - 6y^2 - 3x^2$.

a) Using the First Derivative Test, find all candidates for local extrema of $f$ in the plane.
b) Determine for each of these whether it is a local maximum, a local minimum, or neither, using the Second Partials Test.

Problem 7 Consider the function

$$f(x, y) = x^2y - y + 2$$

on the triangle $T$ with vertices $(0, -6), (0, 6)$, and $(3, 0)$. Find the maximum and minimum value of $f(x, y)$ on $T$, and all points where they are attained.