Problem 1 Consider the function

\[ f(x, y) = 2xy^2. \]

a) Find the gradient \( \nabla f(x, y) \).
b) Find the standard linearization \( L(x, y) \) of \( f \) in the point \( P_0 = (4, 1) \).
c) Approximate \( f(3.997, 1.002) \) using the standard linearization of part b). Use three digits after the decimal point for your answer.

Solution.  a) The gradient is

\[ \nabla f = [2y^2, 4xy]. \]

b) At the point \( P_0 \), the function \( f(x, y) \) has value 8 and its gradient equals \([2, 16]\). So the linearization is

\[ L(x, y) = 8 + 2(x - 4) + 16(y - 1). \]

(1)

c) We use the equation (1) and put in \((x, y) = (3.997, 1.002)\). The result is

\[ f(3.997, 1.002) \approx 8 + 2(-0.003) + 16(0.002) = 8.026. \]

The exact answer is

\[ f(3.997, 1.002) = 8.026007976. \]

Problem 2 The equation

\[ x^2 + 2xy + 5y^2 - z^2 = 4 \]

describes a surface called a one-sheeted hyperboloid.
a) Find a normal vector to this surface in the point \( P = (2, 1, 3) \).
b) Find an equation for the tangent plane to this surface in the point \( P = (2, 1, 3) \).
c) Find ONE point on this surface where the tangent plane is parallel to the \( z \)-axis.
Solution. a) The gradient of \( F(x, y, z) = x^2 + 2xy + 5y^2 - z^2 \) is

\[
\nabla F = [2x + 2y, 2x + 10y, -2z].
\]

In the point \( P \), this equals \([6, 14, -6]\). This vector is normal to the surface in \( P \).
b) Therefore, an equation for the tangent plane can be written as

\[
6(x - 2) + 14(y - 1) - 6(z - 3) = 0.
\]

Above, you can see two views of this surface with the normal vector and a rectangular piece of the tangent plane in \([2, 1, 3]\).
c) The tangent plane is parallel to the \( z \)-axis exactly if the normal \( \nabla F \) is perpendicular to the \( z \)-axis. This happens exactly for \(-2z = 0\), so \( z = 0 \). All points on the surface \( F(x, y, z) = 4 \) with this property are described by \( z = 0 \) and

\[
F(x, y, 0) = x^2 + 2xy + 5y^2 = 4.
\]

Examples are \((2, 0, 0), (-2, 0, 0)\), and \((0, \pm \sqrt{5/4}, 0)\).

Problem 3 Wind sweeps the Great Plains! Suppose that air pressure at point \((x, y)\) of the \( xy \)-plane is given by

\[
P(x, y) = x^2 + 4x - xy - y^2.
\]

Air particles always move in direction of lowest pressure (the direction where pressure decreases the fastest).
a) An air particle is at point \((3, 2)\). Find a unit vector in the direction in which the particle is moving.
b) Find an equation for the tangent line to the level curve of \( P \) through the point \((3, 2)\).
Solution. a) The gradient of \(P\) is
\[
\nabla P = (2x + 4 - y, -x - 2y,)
\]
Evaluate this at (3, 2) to get \(\nabla P(3, 2) = (8, -7)\). Since the particle always moves opposite to \(\nabla P\), the unit vector in its direction is
\[
u = \frac{1}{||\nabla P(3, 2)||} \nabla P(3, 2) = -\frac{1}{\sqrt{113}} (8, -7).
\]
b) The tangent line to the level curve through (3, 2) will be perpendicular to the gradient of \(P\) at that point. So its direction vector must be a multiple of (7, 8), to make a dot product zero with the gradient. A parametric representation of the tangent line could then be
\[
\mathbf{r}(t) = t(7, 8) + [3, 2] = (7t + 3, 8t + 2).
\]
For an equation, we note that the slope has to equal 8/7, so we could write \(\frac{8}{7}(x - 3) + 2\).

Problem 4 Consider the function \(f(x, y) = 3x^2 - xy + y^3\).

a) Using the First Derivative Test, find all candidates for local extrema of \(f\) in the plane.
b) Determine for each of these whether it is a local maximum, a local minimum, or neither, using the Second Partial Test.

Solution. a) The gradient \(\nabla f = (6x - y, -x + 3y^2, )\) vanishes for \(y = 6x\)
and $x = 0$ or $x = 1/108$. The candidates for local extrema are the two points (0, 0), (1/108, 1/18).

b) First calculate $f_{xx} = 6$, $f_{xy} = -1$, $f_{yy} = 6y$ and then $D = 36y - 1$. Then plug in the candidates to get

<table>
<thead>
<tr>
<th>point</th>
<th>$D$</th>
<th>loc. max/min</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>-1</td>
<td>saddle point</td>
</tr>
<tr>
<td>(1/108, 1/18)</td>
<td>2</td>
<td>loc. min. since $f_{xx} &gt; 0$.</td>
</tr>
</tbody>
</table>

Above is a contour plot of $f$, overlaid with quiver plot of $\nabla f$. There is a saddle point at (0, 0) (green) and a local extremum at (1/108, 1/18) (red). Only by looking at the arrows pointing away from the second point can you tell that the extremum is actually a local minimum.

**Problem 5** You are supposed to build an open-top box out of a rectangular bottom and four rectangles attached to each of the sides of the bottom. Use marble for the bottom, at a cost of $20 per square foot. Material for the sides is stainless steel which costs $6 per square foot.

a) Express the cost $C(x, y, z)$ of materials for this box as a function of dimensions $x, y$ for the bottom and $z$ for the height of the box.
b) You need to build a box which has a volume of 360 cubic feet. Find the dimensions of such a box that minimize the cost.

**Solution.** Let $x, y$ be the sides of the bottom and $z$ the height of the box.

$$C(x, y, z) = 20xy + 12(xz + yz).$$

$$V(x, y) = xyz$$

so $z = 360/(xy)$ and the cost is

$$C(x, y) = 20xy + 4320/y + 4320/x.$$ 

The gradient of $C$ is

$$\nabla C(x, y) = \left(20y - 4320/x^2, 20x - 4320/y^2\right).$$

This equals zero exactly if

$$x^2y = xy^2 = 216.$$ 

So all coordinates have to be nonzero, and $x = y = \sqrt[3]{216} = 6$. Then you get $z$ as

$$z = \frac{360}{xy} = 10.$$