15.8 - General Change of Coordinates, eyepatch example

**Example.** We are going to apply a coordinate change to compute the mass of a 2D-plate defined by

\[ 9 \leq x^2 + y^2 \leq 16, \quad 1 \leq y^2 - x^2 \leq 9. \]

and density \( k|x^4 - y^4| \).

We want to define \( u, v \) such that

\[ u = x^2 + y^2, \quad v = y^2 - x^2. \quad (1) \]

So how should we define the functions \( x = g(u, v) \) and \( y = h(u, v) \)? We solve the equations in (1) for \( x \) and \( y \). For \((x, y)\) in the first quadrant,

\[ x = g(u, v) = \sqrt{\frac{u - v}{2}}, \quad y = h(u, v) = \sqrt{\frac{u + v}{2}} \]

and our transformation will only work for points \((u, v)\) where both square roots are defined. But, by symmetry, we can compute the mass of the quarter \( S \) of our region that lies in the first quadrant (colored black in the plot above), and multiply by four afterwards. Before we start, we need to compute the Jacobian

\[ J(u, v) = \begin{vmatrix} 1 & -1 \\ 8\sqrt{u^2 - v^2} & 8\sqrt{u^2 - v^2} \end{vmatrix} = \frac{1}{4\sqrt{u^2 - v^2}}. \]
We also need to compute our limits for \( u, v \). They are easy! \( 9 \leq u \leq 16 \) and \( 1 \leq v \leq 9 \), and \( R \) is a rectangle. This already ensures \( u \geq v \), so \( g(u, v) \) is defined on \( R \). The mass of the lamina is then

\[
m = 4k \int \int_S |x^4 - y^4| \, dx \, dy = 4k \int \int_R \frac{uv}{4\sqrt{u^2 - v^2}} \, dv \, du
\]

\[
= k \int_1^9 v \left[ (u^2 - v^2)^{1/2} \right]_{16}^9 \, dv
\]

\[
= k \int_1^9 v((16^2 - v^2)^{1/2} - (9^2 - v^2)^{1/2}) \, dv
\]

\[
= -\frac{k}{3} \left[ (256 - v^2)^{3/2} - (81 - v^2)^{3/2} \right]_1^9
\]

\[
= \frac{k}{3} (255^{3/2} - 80^{3/2} - 175^{3/2}).
\]

Now that you have made it through this long computation, you should look again at the picture: