Math 165 – worksheet for ch. 5, Integration – solutions

Chapter 5 covers topics including

- Antiderivatives (OK, holdover from chapter 4, but resumed in 5.5)
- Riemann Sums, Sigma Notation (5.2)
- Simple Integrals, connection to Area (5.3)
- FTC I (derivative of accumulation function is integrand) (5.4)
- FTC II (evaluation theorem) (5.4)
- Substitution Method (5.5)
- Area between curves (5.6)

**Problem 1** Solve the initial value problem $y = f(t)$ with

\[
y' = 2t + 5 \cos(\pi t)
\]
\[
y(2) = 18
\]

**Solution** First, find $y$ as an antiderivative of $2t + 5 \cos(\pi t)$,

\[
y = \int 2t + 5 \cos(\pi t) \, dt = t^2 + \frac{5}{\pi} \sin(\pi t) + C.
\]

(you could use a substitution $u = \pi t$ here, with $\frac{du}{\pi} = dt$). Then use the second of the given equations to get

\[
18 = y(2) = 4 + \frac{5}{\pi} \sin(2\pi) + C = 4 + C
\]

so $C = 14$ and

\[
f(t) = t^2 + \frac{5}{\pi} \sin(\pi t) + 14.
\]
**Problem 2**  
a) Express the sum \( S = 1/1 + 1/2 + 1/3 + \cdots + 1/100 \) using Sigma notation.

b) Estimate the sum by comparing it to an integral over \( 1/x \).

**Solution**  
For a), there are several correct answers, here are two:

\[
S = \sum_{k=1}^{100} \frac{1}{k} = \sum_{i=0}^{99} \frac{1}{i+1}.
\]

For b), use that \( S \) is a Riemann sum for the function \( y = 1/x \) on the interval \([1,101]\) split into 100 equal parts, and choosing the left endpoint of each subinterval to evaluate \( 1/x \). So

\[
S \approx \int_1^{101} \frac{1}{x} \, dx = \left[ \ln x \right]_1^{101} = \ln(101) \approx 4.62.
\]

Arguably, this is all we have to do for this problem (on an exam, we’d definitely count this as complete). But we can actually do a bit more: obviously, the rectangles making up the region which has area \( S \), taken together, cover the region under the graph of \( 1/x \), so

\[
\ln(101) \leq S.
\]

We could also consider a Riemann sum for \( \ln x \) where we choose the left endpoint of each of the same 100 intervals from \( x = 1 \) to \( x = 101 \). This will give

\[
\sum_{k=2}^{101} \frac{1}{k} \leq \ln(101).
\]

The sum on the left above equals \( S - 1 + 1/101 \), so

\[
S - 1 + \frac{1}{101} \leq \ln(101).
\]

Recombine the two inequalities to get

\[
\ln(101) \leq S \leq \ln(101) + 1 - \frac{1}{101}.
\]

So the approximation has at most an error of \( 100/101 \approx 0.99 \). It’s not great but in the ballpark.
**Problem 3** Find the integral of the function $f(t)$ graphed below between $x = 0$ and $x = 13$.

![Graph of the function $f(t)$](image)

**Solution** Cut the region into two triangles from $x = 0$ to $x = 1.5$ and $x = 1.5$ to $x = 3$, then a rectangle from $x = 3$ to $x = 10$ with a semicircle on top that has diameter 4, and another triangle from $x = 10$ to $x = 13$. The first two triangles have area $9/4$ each but they cancel each other out (for the integral, areas under the $x$-axis count as negative). The rectangle has area 21, the semicircle has area $2\pi$ and the last triangle has area $9/2$. The final answer is then

$$\int_{0}^{13} f(t) \, dt = 25.5 + 2\pi.$$

**Problem 4** Find the derivative of the following functions.

a) $f(z) = \int_{5}^{z} \frac{e^{t}}{2 + \sin t} \, dt$,

b) $g(u) = \int_{u}^{4} \frac{1}{t(1 + \cos t)} \, dt$,

c) $h(y) = \sin \left( \int_{3}^{y} \sin(t^2) \, dt \right)$,

d) $L(x) = \int_{\cos x}^{\sin x} \frac{t^3}{t^2 + 1} \, dt$. 
Solution  This is about the Fundamental Theorem of Calculus, part I. Note that we cannot compute these functions themselves with paper and pencil – luckily, we do not have to.

\[ f'(z) = \frac{e^z}{2 + \sin z} \]
\[ g'(u) = -\frac{1}{u(1 + \cos u)} \]
\[ h'(y) = \cos \left( \int_3^y \sin(t^2) \, dt \right) \sin(y^2) \]
\[ L'(x) = \frac{\sin^3 x \cos x}{\sin^2 x + 1} + \frac{\cos^3 \sin x}{\cos^2 + 1} \]

Problem 5  a) Find the area under the graph of \( x^3 e^{-x^4} \) between \( x = 0 \) and \( x = 5 \).
b) Find the average value of \( x^3 e^{-x^4} \) between \( x = 0 \) and \( x = 5 \).

Solution  a) We use \( u = x^4 \), so \( \frac{1}{4} \, du = x^3 \, dx \). Then

\[ A = \int_0^5 x^3 e^{-x^4} \, dx = \frac{1}{4} \int_0^{5^4} e^{-u} \, du = \frac{1}{4} \left[ -e^{-u} \right]_0^{625} = \frac{e^{625} - 1}{4}. \]

b) Just divide your answer from a) by the length of the interval \([0, 5]\), which is 5. So the average is

\[ \frac{e^{625} - 1}{4}. \]

Problem 6  If \( f(x) \) is continuous on \([0, a]\), find the integral

\[ M = \int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx. \]

Try making the substitution \( u = a - x \), and adding the resulting integral to \( M \).
Solution  So we make the substitution $u = a - x$. We get $du = -dx$ and new limits $u = a$ and $u = 0$. The value of the integral stays the same, so

$$M = -\int_a^0 \frac{f(a-u)}{f(a-u) + f(u)} du = \int_0^a \frac{f(a-x)}{f(x) + f(a-x)} dx.$$

Then we add this to the original integral to get

$$2M = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx + \int_0^a \frac{f(x-a)}{f(x) + f(a-x)} dx$$

$$= \int_0^a \frac{f(x) + f(x-a)}{f(x) + f(x-a)} dx = a.$$

So $M = a/2$.

Problem 7 Find the total area enclosed by the curves

$$y = x^3 - x + 4$$
$$y = x^3 - x + x^2$$

Solution

Where do they meet? solve

$$x^3 - x + 4 = x^3 - x + x^2$$

which gives $x^2 - 4 = 0$, so at $x = \pm 2$. A graph reveals that $x^3 - x + 4$ is on top for $x$ in $[-2, 2]$, so area equals

$$A = \int_{-2}^2 4 - x^2 \, dx = 2 \left[ 4x - \frac{x^3}{3} \right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}.$$