Math 165 - Practice Exam 1 - solutions

Problem 1 Evaluate the following limits.

a) \[ \lim_{x \to 0} \cos(\pi(x + 1)) = -1 \]

b) \[ \lim_{y \to 0} \frac{\tan(4y^2)}{3y^2} = \frac{4}{3} \]

c) \[ \lim_{t \to 1} \frac{\cos(2\pi t) - 1}{t - 1} = 0 \]

In c), you may use the addition formula of the cosine,

\[ \cos(a + b) = \cos a \cos b - \sin a \sin b \]

Solution In a), we only have to substitute \( x = 0 \).
In b), we use the intermediate variable \( u = 4y^2 \), then the special trig limit \( \sin(u)/u \), as in

\[ \lim_{y \to 0} \frac{\sin(4y^2)}{4y^2} = \lim_{u \to 0} \frac{\sin u}{u} = 1. \]

Then bring this into play via

\[ \lim_{y \to 0} \frac{(\tan(4y^2))}{3y^2} = \lim_{y \to 0} \frac{4 \sin(4y^2)}{3 \cdot 4y^2 \cos(4y^2)} = \frac{4}{3} \cdot \lim_{y \to 0} \frac{4}{3 \cos(4y^2)} = \frac{4}{3}. \]

(the last limit can be done by substitution).
In c), we use the intermediate variable \( u = t - 1 \) and the addition formula of the cosine, with \( a = 2\pi u \) and \( b = 2\pi \), so

\[ \cos(2\pi t) = \cos(2\pi(u + 1)) = \cos(2\pi u) \]

and then the special trig limit for the cosine.
**Problem 2** Evaluate the limits, if they exist.

a) \[ \lim_{t \to \infty} \frac{5t^4 - 2t^3 + 9}{2t^4 + 100} = \frac{5}{2} \]

b) \[ \lim_{x \to \infty} \frac{3 - 2x + 4x^3}{3x^2 + 2x} = \infty \]

c) \[ \lim_{u \to 3} \frac{5 - 2u}{u^2 - 9} \text{ does not exist} \]

d) \[ \lim_{v \to 3^+} \frac{5 - 2v}{v^2 - 9} = -\infty \]

e) \[ \lim_{w \to \infty} \frac{2w^4 - \sin(w)}{w^5 + 1} = 0. \]

**Solution** For a), we have a rational function with the same degree (four) in numerator and denominator, so we can factor out \( t^4 \) and then apply rules.

\[ \lim_{t \to \infty} \frac{5 - 2t^{-1} + 9t^{-4}}{2 + 100t^{-4}} = \frac{5}{2} \]

because all those powers of \( t \) with negative exponent have limit zero.

For b),

\[ \lim_{x \to \infty} \frac{3 - 2x + 4x^3}{3x^2 + 2x} = \infty \]

because the numerator in this rational function has the higher degree (3 versus 2).

For c), the right-handed limit is \(-\infty\) and the left-handed limit is \(\infty\) (similar to d), see below). So the two-sided limit does not exist.

For d), the limit is \(-\infty\) because the limit of the numerator is \(-1\) and the limit of the denominator is 0, but the fraction is negative (numerator negative, denominator positive) for \( v \) approaching 3 from the right.

For e), the limit is zero because this is a rational function with a numerator of lower degree than the denominator (apart from the \( \sin(w) \) term which is dominated by any power of \( w \)).

**Problem 3** The following functions are all defined to be 0 for \( x = 0 \). The definitions below apply to \( x \neq 0 \). Which of these are continuous at \( x = 0 \), which ones are not? Give a reason for every one of your decisions.

A: \( 2|x| - 2 \)  B: \( \frac{\sin(x)}{\sqrt{|x|}} \)  C: \( \sin\left(\frac{1}{x}\right) \)  D: \( \tan(x) \)
Solution  

A is not cont. at 0 because its limit at \( x = 0 \) is \(-2\), not 0.

B is continuous because

\[
\frac{\sin x}{\sqrt{|x|}} = \frac{\sin x}{x} \cdot \frac{x}{\sqrt{|x|}},
\]

the first factor has limit 1 and the second has limit 0 since it has absolute value \( \sqrt{|x|} \).

C is not cont. because it oscillates between \(-1\) and \(1\), no matter how small \( x \) is, so its limit at 0 does not exist.

D is cont. because the sine has limit 0 at 0, the cosine in the denominator has limit 1.

**Problem 4**  
Suppose the functions \( f(x) \) and \( g(x) \) are continuous everywhere and have these values: \( f(0) = 1 \), \( f(1) = 5 \), \( f(2) = 4 \), \( g(0) = 2 \), \( g(2) = 1 \), \( g(4) = 3 \). Find \( \lim_{x \to 2} f(x) + g(x) \) and \( \lim_{x \to 2} f(g(x)) \).

Solution

\[
\lim_{x \to 2} g(x) + f(x) = 1 + 4 = 5
\]

\[
\lim_{x \to 2} f(g(x)) = f(1) = 5
\]

**Problem 5**  
a) Find the instantaneous rate of change of

\[ f(x) = 2x^2 + 5x \]

at \( x = -2 \). Use a limit computation, no rules for derivatives that are not covered in Chapter 2.

b) Find an equation for the tangent line to the graph of \( f(x) \). Make a sketch of the graph and this tangent line, using the values

| \( x \) | \(-3\) | \(-2\) | \(-1\) | \(0\) | \(1\) |
| \( f(x) \) | \(3\) | \(-2\) | \(-3\) | \(0\) | \(7\) |
Solution  

a) First, the instantaneous rate of change is 

\[ m = \lim_{h \to 0} \frac{1}{h} [2(x + h)^2 + 5(x + h) - (2x^2 + 5x)] \]

\[ = \lim_{h \to 0} \frac{1}{h} [4xh + h^2 + 5h] = \lim_{h \to 0} [4x + h + 5] = 4x + 5. \]

You can substitute \( x = -2 \) at any time you like. Your final answer should then be \( m = -8 + 5 = -3. \)

b) Therefore the tangent line has equation \( y = -3(x + 2) + f(-2) = -3(x + 2) - 2. \)

Problem 6  Evaluate the following limit.

\[ \lim_{t \to 0^+} \frac{\sqrt{\sin(4t^2)}}{t} \]

Solution  There are several ways to solve this. One way is to rewrite the fraction as 

\[ \sqrt{\frac{\sin(4t^2)}{t^2}} \]

and then compute the limit of the argument of the square root, which is 

\[ \lim_{t \to 0} \frac{\sin(4t^2)}{t^2} = \lim_{u \to 0} \frac{\sin 4u}{u} = 4. \]

The square root of this quantity has then limit \( \sqrt{4} = 2 \), again by the Main Limit Theorem.
Problem 7 Find the limit
\[ L = \lim_{x \to 0} \frac{\sqrt{x^2 + x + 1} - \sqrt{x^2 - 3x + 1}}{x}. \]

Solution Substitution does not work, so we try algebra.
\[ \sqrt{x^2 + x + 1} - \sqrt{x^2 - 3x + 1} = \frac{4x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - 3x + 1}} \]
which we use for the limit in question,
\[ L = \lim_{x \to 0} \frac{4}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - 3x + 1}} = 2. \]
The last limit can be done by substitution (Limit Law for quotients) since the limit of the denominator is nonzero.

Problem 8 One night, a car drives on the road given by \( y = x^2 - 4x \) between \( x = -5 \) and \( x = 3 \). Find the point(s) \( x = a \) at which its headlights shine on the point \( P(2, 0) \). Make a sketch first! Show your working.

Solution First, the derivative of \( x^2 - 4x \) at \( x = a \) is \( 2a - 4 \). So the tangent line at \( x = a \) has equation
\[ y = (2a - 4)(x - a) + (a^2 - 4a). \]
To pass through the point \( (2, 0) \), we need to satisfy the condition \( y = 0 \) when \( x = 2 \),
\[ 0 = (2a - 4)(2 - a) + a^2 - 4a \]
which simplifies to \( 0 = -a^2 + 4a - 8 \) which has no solution since \( 4^2 - 32 < 0 \). So the answer is the headlights never shine on that point (Interesting bonus question: what is the region in the plane of all points that are never touched by the car’s headlights? answer: all the points above the parabola).

As an example of a similar question where there would be a solution, the sketch below shows those points \( x = a \) where the tangent line passes through \( (2, -8) \), leading to
\[ -8 = (2a - 4)(2 - a) + a^2 - 4a, \]
namely $a = 0$ and $a = 4$. If the car moves from left to right, say, then of course the headlights would not shine backwards, so only $a = 0$ is such a point.