Problem 1  A car has position at time $t$ given by

$$s(t) = 36t - 12t^2$$

(distance in meters, time in seconds).
(a) Find the velocity of the car at time $t$.
(b) Find the acceleration of the car at time $t$.
(c) At what time $t$ does the car come to a stop (has velocity 0)?
(d) What is the position of this car at the time in (c)? (note - this is how far
away from a person in the road the driver needs to brake (= apply negative
acceleration) in order to avoid hitting the person).

Solution.  (a) $v(t) = s'(t) = 36 - 24t$.
(b) $a(t) = s''(t) = -24$.
(c) Solve $v(t) = 0$ to get $t = 36/24 = 1.5$ (seconds).
(d) $s(1.5) = 54 - 27 = 27$ (in meters).

Problem 2  Find the absolute maximum and absolute minimum values of
the function

$$f(x) = x^4 - 4x^3 + 6x^2 + 1$$
on the interval $[-2, 2]$. Indicate at which $x$-values each of these extrema
occur.

Solution.  We need to calculate

$$f'(x) = 4x^3 - 12x^2 + 12x$$

and solve $f'(x) = 0$. Factorize $f'(x)$ as

$$f'(x) = 4x(x^2 - 3x + 3)$$

is zero at $x = 0$ only (use the quadratic formula or complete the square to
decide that $x^2 - 3x + 3 = 0$ has no solution). So we have only three candidates
$x = -2, 0, 2$, and the table of values of $f(x)$ at these numbers is

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$0$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>73</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>
and therefore the maximum of $f(x)$ on $[-2, 2]$ is 73, occurring at $x = -2$. The minimum is 1, occurring at $x = 0$.

**Problem 3** Suppose the demand $x$ (= number of units sold) of Halloween Slime Cookies$^\text{sm}$ is linked to unit price $p$ by

$$p = \sqrt{500 - x^2}$$

(a) Find the revenue $R(p)$ as a function of $p$.
(b) Find the price $p$ which gives you maximal revenue.

**Solution.** (a) First, we solve the demand equation for $x$ to get

$$x = \sqrt{500 - p^2}$$

and then use $R = xp$ to get

$$R(p) = p\sqrt{500 - p^2}.$$  

(b) Obviously, only $0 \leq p \leq \sqrt{500}$ are legitimate values for $p$. There, we differentiate

$$R'(p) = \frac{-2p}{2\sqrt{500 - p^2}} = \frac{500 - p^2}{\sqrt{500 - p^2}}.$$  

Solving $R' = 0$ gives

$$500 - 2p^2 = 0$$

with solutions $p = \pm\sqrt{250} = \pm5\sqrt{10}$. Discarding the negative answer, the critical numbers (including endpoints) are $p = 0, \sqrt{250}, \sqrt{500}$. Note that $p = \sqrt{500}$ is also a point where $R(p)$ is not differentiable (has a vertical tangent line), but we had it on the list as endpoint anyway. Clearly, $R(0) = R(\sqrt{500}) = 0$ is not the maximum, so $p = \sqrt{250}$ will give maximal revenue (we could have made a table, too).

**Problem 4** Suppose the revenue $R(x)$ for selling $x$ boxes of Halloween Slime Cookies$^\text{sm}$ is actually

$$R(x) = 6x - 0.03x^2 + 0.001x^3.$$  

Find the number $x$ in $[0, 1000]$ (of boxes of cookies to sell) so that the revenue is maximal.

**Solution.** First,
This is always positive, \( R'(x) = 0 \) has no solution. So \( R(x) \) is always increasing, and the maximum revenue in the given \( x \)-interval occurs at \( x = 1000 \).

**Problem 5**  If each edge of a cube is increasing at a rate of 3 centimeters per minute, how fast is the volume increasing when \( x \), the length of an edge, is 15 centimeters long?

**Solution.**  Let \( V \) be the volume of the cube, so \( V = x^3 \) at all times \( t \). Therefore

\[
V'(t) = 3[x(t)]^2x'(t)
\]

and substituting \( x = 15 \) and \( x' = 3 \), we get \( V'(t) = 2025 \) (in cubic centimeters per minute).

**Problem 6**  Suppose gross domestic product (GDP) \( G \) and population \( P \) of a country are related by the equation

\[
G^2 - 0.3GP + P^{2/3} = 17.
\]

Both \( G \) and \( P \) are functions of time \( t \). At a time when \( G = 5 \) and \( P = 8 \) (in billions $ and millions of people, respectively), the GDP grows at a rate of \( G'(t) = 0.06 \) billion $ per year. Use related rates to find the corresponding rate of change for \( P \). Include units in your answer.

**Solution.**  We can differentiate both sides, like in the previous problem, using the Chain Rule.

\[
2GG' - 0.3(G'P + GP') + \frac{2P'}{3P^{1/3}} = 0
\]

Then we substitute \( G = 5 \), \( P = 8 \), and \( G' = 0.06 \) to get

\[
0.6 - 0.3(0.48 + 5P') + \frac{2P'}{6} = 0.
\]

Solving this for \( P' \) (multiply by 3, collect all terms with \( P' \) on the right side, divide by 3.5) gives

\[
P' = \frac{1.368}{3.5} \approx 0.391
\]

(in millions of people per year).
Problem 7  Two quantities $x, y$ are related by

$$y^3 + xy = x^3 - x^2 - 1.$$ 

Suppose that $y$ is a function of $x$, ie $y = f(x)$ and that for $x = 2$, $y = 1$.
(a) Find $\frac{dy}{dx}$ at $x = 2$ using implicit differentiation.
(b) Find an equation for the tangent line of the graph of $f(x)$ at $x = 2$. Use the form $y = m(x - 2) + b$.
(c) What $y$-value do you get for the tangent line equation from part (b) at $x = 2.03$?
Not part of this problem: this is an excellent approximation of $f(2.03)$, and this works for all $x$-values close to 2. Useful because the given equation $y^3 + xy = x^3 - x^2 - 1$ is hard to solve for $y$. See the graph of $f(x)$ and the tangent line below.

Solution.  (a) Differentiate both sides:

$$3y^2 y' + y + xy' = 3x^2 - 2x$$

then collect all terms with $y'$ on the left side, factor out $y'$ and solve for $y'$:

$$y'(x) = \frac{3x^2 - 2x - y}{3y^2 + x}$$

(b) Use part (a) and plug in $x = 2$, $y = 1$ into the formula for $y'$ to get

$$y'(2) = \frac{12 - 4 - 1}{3 + 2} = \frac{7}{5}.$$
Then the desired tangent line has slope $7/5$, passes through $(2, 1)$. It therefore has a point-slope equation

$$y = \frac{7}{5}(x - 2) + 1.$$ 

(c) Simply substitute $x = 2.03$ into the tangent line equation from part (b) to get

$$y = \frac{7}{5}(2.03 - 2) + 1 = 1.042.$$

Problem 8  Find the indefinite integrals (= family of all antiderivatives) of

(a) $f(x) = 15x^4 - 8x + 1$
(b) $g(u) = 2u^{3/4} - 3$
(c) $h(t) = 12(3t^4 - 5)^8t^3$

Solution.  Make sure to keep the same variable as in the problem, and to add on the integration constant at the end. For (c), use the substitution $u = 3t^4 - 5$.

$$\begin{align*}
(a) & \int f(x) \, dx = 3x^5 - 4x^2 + x + C \\
(b) & \int g(u) \, du = \frac{8}{7}u^{7/4} - 3u + C \\
(c) & \int h(t) \, dt = \frac{1}{9}(3t^4 - 5)^9 + C
\end{align*}$$

Problem 9  Suppose the marginal productivity in building $x$ luxury cars is

$$P'(x) = 2e^{-0.1x}.$$ 

(a) Solve an indefinite integral to find $P(x)$ up to a constant.
(b) Given that $P(0) = 0$, find $P(x)$.

Solution.  (a) We use our integration formulas from 7.1 to get

$$P(x) = \int 2e^{-0.1x} \, dx = -20e^{-0.1x} + C.$$ 

(b) From $P(0) = 0$, substituting into the preceding equation, we get

$$0 = -20 + C$$
so $C = 20$ and

$$P(x) = 20 - 20e^{-0.1x}.$$