Lec. 10: 2-4-09

1. Cons. of momentum
2. Constitutive relations
3. Navier-Stokes eqns

Cons. of momentum

\[ \text{ROE 2-mom} = \text{net flux in} + \text{body forces} + \text{net stress} \]

\[ \text{ROE 2-mom} = \frac{\partial}{\partial t} (p \omega x a y z) \]

Net flux in = \(- \Delta (p w w) a y z - \Delta (p w w) a x y \)

Body force = \(- p g a x y a z \)

Net stress = \((c_{33} | c_{33} a x x - c_{33} | c_{33} a y y) a y z \)
\[ \quad + (c_{33} | c_{33} a x x - c_{33} | c_{33} a y y) a x y \]

\[ \frac{\partial}{\partial t} (p \omega x a y z) = - \Delta (p w w) a y z - \Delta (p w w) a x y \]
\[ \quad - p g a x y a z + (c_{33} | c_{33} a x x - c_{33} | c_{33} a y y) a y z \]
\[ \quad + (c_{33} | c_{33} a x x - c_{33} | c_{33} a y y) a x y \]

\[ \frac{\partial}{\partial t} (p w w) = - \frac{\Delta (p w w)}{\Delta x} - \frac{\Delta (p w w)}{\Delta z} - p g + \frac{c_{33} | c_{33} a x x - c_{33} | c_{33} a y y}{\Delta x} \]
\[ \quad + \frac{c_{33} | c_{33} a x x - c_{33} | c_{33} a y y}{\Delta z} \]

\( \Delta x, \Delta z \to 0 \)

\[ \frac{\partial}{\partial t} (p w w) + \frac{\partial}{\partial x} (p w w) + \frac{\partial}{\partial z} (p w w) = - p g + \frac{2 c_{33}}{\Delta x} + \frac{2 c_{33}}{\Delta z} \]
\[
\frac{\partial}{\partial t} (\rho w) + \frac{\partial}{\partial x} (\rho w u) + \frac{\partial}{\partial z} (\rho w w) = -\rho g + \frac{\partial p x}{\partial x} + \frac{\partial p z}{\partial z}
\]

**CLDT - Conservation Law Derivation Trick**

Student error: To assume \( \rho = \text{constant} \)

\( \rightarrow \) Not necessary. Use CLDT - cons. of mass

Product rule on left side

LHS = \( \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) \)

\[+ w \left( \frac{\partial p x}{\partial t} + \frac{\partial p x}{\partial x} + \frac{\partial p z}{\partial x} \right) \]

= 0 by cons. of mass (in 2D)
Summarize: Incompressible fluid \((x, y, z)\)

\[ \text{mass: } \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]

\[ z\text{-mom: } \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\rho g + \frac{\partial P}{\partial x} + \frac{\partial T}{\partial z} \]

3D:

\[ \text{mass: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[ z\text{-mom: } \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\rho g + \frac{\partial P}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} \]

Equations: 4 (mass, x, y, z mom)

Unknowns: 12 \((u, v, w, \tau_{xx}, \tau_{xy}, \tau_{yx}, \tau_{yy}, \tau_{yz}, \tau_{zx}, \tau_{zy}, \tau_{zz})\)

More unknowns than equations \(\rightarrow\) System is not closed

\(\rightarrow\) Closure problem

Cons. of angular momentum \(\rightarrow\) \(\tau_{xx} = \tau_{yy} = \tau_{zz}, \text{ etc.} \)

\(\rightarrow\) 9 unknowns

Need other relations: In our viscous slot flow, we related stress to the velocity gradient
**Constitutive relations**

**Static fluid**
- Shear stresses are zero
- Normal stress is due to pressure
  
  **Isotropic**

\[ \tau_{xx} = \tau_{yy} = \tau_{zz} = -p \]

**z-mom:**
\[ p \left( \frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial x} + \frac{\partial w}{\partial z} \right) = -\rho g + \frac{\partial p}{\partial x} + \frac{\partial (\rho v)}{\partial z} \]

**static:**
\[ \nu = 0, \quad \tau_{zz} = -p \]

\[ 0 = -\rho g - \frac{\partial p}{\partial z} \]

\[ \frac{\partial p}{\partial z} = -\rho g \]

\[ p = -\rho g z + C \]

\[ p(H) = p_0 \Rightarrow p_0 = -\rho g H + C \]

\[ p = p_0 + \rho g (H-z) \]