6. Environmental flows can have varying density but still be incompressible.
   a. Suppose the density is given by
      \[ \rho = \rho_0 + \rho_a \sin(mz - \omega t), \]
      where \( \rho_0, \rho_a, m, \) and \( \omega \) are constants. What velocity field will make this flow incompressible?
   b. True or false: All incompressible flows have constant density.

7. A flow has velocity \((u, v, w) = (axyz, by^2z, cx^2y)\). For what value of \( b \) is the flow incompressible?

8. A piston compresses a gas in a chamber in one dimension. The initial density, before the piston starts moving, is \( \rho_0 \), and the piston velocity is \( V \). The velocity \( u \) of the air varies linearly between \( V \) at the piston and zero at the wall. At \( t = 0 \), the chamber length is \( L_0 \). Compute the density as a function of time. Provide three reasons why your result is plausible. (Hint: Remember that the length \( L \) of the chamber is a function of time.)
Lake Osoboxy is a rectangular lake whose background temperature profile is linear with a gradient $dT/dz$ of 0.5°C/m. The lake is 1000 m long, 250 m wide, and 20 m deep. When wind blows on the lake, long, standing internal waves, or seiches, can form. To measure these motions, two chains of thermistors, which record temperature as a function of time, are installed 230 m and 680 m from one shore. The data files on the website give the depths (in m) of the temperature readings, the times (in s) of the measurements, and the temperatures in degrees Celsius. (Only part b below is required.)

a. Isotherms are lines of constant temperature. If the temperature fluctuation is given by

$$T = -w_0 \frac{dT}{dz} \cos kx \sin mz \cos \omega t,$$

where $w_0$ is an amplitude, $\omega$ is the wave frequency, and $k$ and $m$ are the horizontal and vertical wavenumbers, then the isotherm displacement $\zeta = T/(dT/dz)$ is

$$\zeta = -\frac{w_0}{\omega} \cos kx \sin mz \cos \omega t.$$

Show that the vertical velocity is

$$w = w_0 \cos kx \sin mz \sin \omega t.$$

*b. Show that the horizontal velocity is

$$u = -w_0 \frac{m}{k} \sin kx \sin mz \sin \omega t.$$

c. Now try to extract properties of the waves from temperature measurements with the following steps. First, compute the background temperature profile $T(z)$ by averaging over an appropriate period in time. Compute the buoyancy frequency

$$N = \left(g \alpha \frac{dT}{dz}\right)^{1/2}.$$

d. Determine the vertical wavenumber $m$. One way to do this is to plot the temperature perturbation $T'(z, t) = T - \bar{T}$ for a particular time at one of the stations.

e. Determine the radian frequency $\omega$ and period $2\pi/\omega$ of the waves by plotting the temperature vs. time for one thermistor.

f. The frequency $\omega$ and wavenumbers $k$ and $m$ are related by

$$\omega = N \frac{k}{(k^2 + m^2)^{1/2}}.$$

Compute the wavenumber $k$ and use measurements from both thermistor chains to verify that it is correct.

g. Determine $w_0$, the amplitude of the vertical velocity fluctuations.
10. In class we showed that the rate of angular strain is related to velocity gradients like \( \partial u/\partial y \) and \( \partial v/\partial x \)—that is, derivatives in a direction normal to the velocity component. In this problem you will examine the role of derivatives in the same direction as the velocity component.

a. Consider a fluid element in a flow \( u(x) \). Show that the rate of linear strain is \( \partial u/\partial x \).

b. What do you think the divergence of the velocity (i.e., \( \nabla \cdot u \)) represents?

c. What does the discussion in part b imply for fluid elements in an incompressible flow?

\[ \begin{array}{c}
\text{u(x)} \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\hline
\text{time} = t \\
\text{time} = t + \Delta t
\end{array} \]

11. Explain why the pressure gradient—not the pressure—appears in the Navier-Stokes equations.

*12. In class we considered the flow \( (u, v) = (\Gamma x, -\Gamma y) \). Assume that the fluid has a dynamic viscosity \( \mu \).

a. Sketch streamlines, which are lines tangent to the velocity vector. (You can compute and plot the streamlines, but I want you simply to sketch them.)

b. What might this flow represent? In other words, if your sketch in part a revealed sinusoidal streamlines, you might say that the flow represents flow over sand ripples.

c. Compute the pressure if \( p = p_0 \) at \( x = y = 0 \).

d. Sketch isobars (lines of constant pressure) along with streamlines.

e. Explain in a way that a non-technical person can understand that the relationship between your sketched pressure and velocity fields makes sense.

f. Compute the entire stress tensor.

*13. Compute and plot the shear stress distributions for the following flows of fluid with dynamic viscosity \( \mu \) in a slot with separation \( H \):

a. Flow driven by a pressure gradient \( dp/dx \).

b. Flow driven by the wall at \( y = H \) moving at speed \( U_0 \)

c. Explain the plots. In particular, explain the value of the shear stress on the walls in part a.