36. If \((X, \langle \cdot, \cdot \rangle)\) is an inner product space show that
\[
\langle x, y \rangle = \frac{1}{4} \left( ||x + y||^2 - ||x - y||^2 + i ||x + iy||^2 - i ||x - iy||^2 \right)
\]
Thus, in any normed linear space, there can exist at most one inner product giving rise to the norm.

37. Let \(\Omega \subset \mathbb{R}^n\), \(\rho\) be a measurable function on \(\Omega\), and \(\rho(x) > 0\) a.e. on \(\Omega\). Let \(X\) denote the set of measurable functions \(u\) for which \(\int_\Omega |u(x)|^2 \rho(x) \, dx\) is finite. We can then define the weighted inner product
\[
\langle u, v \rangle_\rho = \int_\Omega u(x) \overline{v(x)} \rho(x) \, dx
\]
and corresponding norm \(||u||_\rho = \sqrt{\langle u, u \rangle_\rho}\) on \(X\). The resulting inner product space is complete, often denoted \(L^2_\rho(\Omega)\). (As in the case of \(\rho(x) = 1\) we regard any two functions which agree a.e. as being the same element of \(X\), so \(X\) is again really a set of equivalence classes.)

   a) Verify that all of the inner product axioms are satisfied.

   b) Suppose that there exist constants \(C_1, C_2\) such that \(0 < C_1 \leq \rho(x) \leq C_2\) a.e. Show that \(u_n \to u\) in \(L^2_\rho(\Omega)\) if and only if \(u_n \to u\) in \(L^2(\Omega)\).

38. Find the Fourier series \(\sum_{n=-\infty}^{\infty} c_n e^{inx}\) for the function \(f(x) = x\) on \((-\pi, \pi)\). Use some sort of computer graphics to plot a few of the partial sums of this series on the interval \([-3\pi, 3\pi]\).

39. Use the Fourier series in problem 38 to find the exact value of the series
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}
\]

40. (Exercise 7.2a in text) If \(f \in C(\mathbb{T})\) and \(S_N\) denotes the \(N\)'th partial sum of its Fourier series, show that \(S_N = D_N * f\) where \(D_N\) is the so-called Dirichlet kernel
\[
D_N(x) = \frac{1}{2\pi} \frac{\sin[(N+1/2)x]}{\sin(x/2)}
\]
Produce a sketch of \(D_N\), either by hand or by computer, for some reasonably large value of \(N\).