21. Is \(d(x, y) = |x - y|^2\) a metric on \(\mathbb{R}\)? What about \(d(x, y) = \sqrt{|x - y|}\)?

22. (Exercise 1.15 in text) Prove that a closed subset of a compact set in a metric space is also compact.

23. If \(\Omega \subset \mathbb{R}^n\) is bounded, show that \(L^p(\Omega) \subset L^q(\Omega)\) whenever \(p > q\). (Hint: Use the Hölder inequality). Show this may be false if \(\Omega\) is not bounded.

24. Let \((X, d)\) be a metric space, \(A \subset X\) be nonempty and define the distance from a point \(x\) to the set \(A\) to be

\[d(x, A) = \inf_{y \in A} d(x, y)\]

(see page 11 in text)

a) Show that \(|d(x, A) - d(y, A)| \leq d(x, y)\) for \(x, y \in X\) (i.e. \(x \to d(x, A)\) is Lipschitz continuous with Lipschitz constant 1).

b) Assume \(A\) is closed. Show that \(d(x, A) = 0\) if and only if \(x \in A\).

c) Assume \(A\) is compact. Show that for any \(x \in X\) there exists \(z \in A\) such that \(d(x, A) = d(x, z)\).

25. (Exercise 1.16 in text) Suppose that \(F\) is closed and \(G\) is open in a metric space \((X, d)\) and \(F \subset G\). Show that there exists a continuous function \(f : X \to \mathbb{R}\) such that

i) \(0 \leq f(x) \leq 1\) for all \(x \in X\).

ii) \(f(x) = 1\) for \(x \in F\).

iii) \(f(x) = 0\) for \(x \in G^c\).

Hint: Consider

\[f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}\]