16. Let \( u_j(t) = t^{\lambda_j} \) where \( \lambda_1, \ldots, \lambda_n \) are arbitrary unequal real numbers. Show that \( \{u_1 \ldots u_n\} \) are linearly independent functions on any interval \((a, b) \subset \mathbb{R}\). (Suggestion: If \( \sum_{j=1}^n \alpha_j t^{\lambda_j} = 0 \), divide by \( t^{\lambda_1} \) and differentiate.)

17. A side condition for a differential equation is homogeneous if whenever two functions satisfy the side condition then so does any linear combination of the two functions. For example the Dirichlet type boundary condition \( u = 0 \) for \( x \in \partial \Omega \) is homogeneous.

Let \( Lu = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha u \) denote any linear differential operator. Show that the set of functions satisfying \( Lu = 0 \) and any homogeneous side conditions is a vector space.

18. Consider the differential equation \( u'' + u = 0 \) on the interval \((0, \pi)\). What is the dimension of the vector space of solutions which satisfy the homogeneous boundary conditions a) \( u(0) = u(\pi) \), and b) \( u(0) = u(\pi) = 0 \). Repeat the question if the interval \((0, \pi)\) is replaced by \((0, 1)\).

19. If \((X, d_X), (Y, d_Y)\) are metric spaces, show that the Cartesian product

\[
Z = X \times Y = \{(x, y) : x \in X, y \in Y\}
\]

is a metric space with distance function

\[
d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)
\]

20. Show by examples that neither of the following conditions concerning a sequence of nonnegative functions \( \{f_n\}_{n=1}^\infty \) implies the other one.

(i) \( \lim_{n \to \infty} f_n(x) = 0 \) for all \( x \in [0, 1] \)

(ii) \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0 \)