11. In the two dimensional case, solutions of Laplace’s equation $\Delta u = 0$ may also be found by means of analytic function theory. Recall that if $z = x + iy$ then a function $f(z)$ is analytic in an open set $\Omega$ if $f'(z)$ exists at every point of $\Omega$. If we think of $f = u + iv$ and $u, v$ as functions of $x, y$ then $u = u(x, y), v = v(x, y)$ must satisfy the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$. Show in this case that $u, v$ are also solutions of Laplace’s equation. Find $u, v$ if $f(z) = z^3$ and $f(z) = e^{z^2}$.

12. Let $\Omega$ be the rectangle $[0, a] \times [0, b]$ in $\mathbb{R}^2$. Find all possible product solutions $u(x, y, t) = \phi(t)\psi(x)\zeta(y)$ satisfying

$$u_t - \Delta u = 0 \quad (x, y) \in \Omega \quad t > 0$$
$$u(x, y, t) = 0 \quad (x, y) \in \partial\Omega \quad t > 0$$

13. Find a solution of the Dirichlet problem for $u = u(x, y)$ in the unit disc $\Omega = \{(x, y) : x^2 + y^2 < 1\}$,

$$\Delta u = 1 \quad (x, y) \in \Omega \quad u(x, y) = 0 \quad (x, y) \in \partial\Omega$$

(Suggestion: look for a solution in the form $u = u(r)$ and recall problem #10.)

14. Derive the ‘four point property’

$$u(x, t) + u(x + h - k, t + h + k) = u(x + h, t + h) + u(x - k, t + k)$$

if $u$ is a solution of the wave equation $u_{tt} - u_{xx} = 0$ in a domain $\Omega \subset \mathbb{R}^2$ containing the tilted rectangle with vertices at $(x, t), (x + h - k, t + h + k), (x + h, t + h), (x - k, t + k)$.

15. In the Dirichlet problem for the wave equation

$$u_{tt} - u_{xx} = 0 \quad 0 < x < 1 \quad 0 < t < 1$$
$$u(0, t) = u(1, t) = 0 \quad 0 < t < 1$$
$$u(x, 0) = 0 \quad u(x, 1) = f(x) \quad 0 < x < 1$$

show that neither existence nor uniqueness holds. (Hint: For the non-existence part, use problem 14 to find an $f$ for which no solution exists.)