56. a) If $f \in L^1(\mathbb{R}^N)$ and $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$, find a relationship between $\hat{f}_\lambda$ and $\hat{f}$.

b) If $f \in L^1(\mathbb{R}^N)$ and $f_h(x) = f(x-h)$ for $h \in \mathbb{R}^N$, find a relationship between $\hat{f}_h$ and $\hat{f}$.

57. Show that

$$\int_{\mathbb{R}^N} \phi(x) \overline{\psi(x)} \, dx = \int_{\mathbb{R}^N} \hat{\phi}(x) \overline{\hat{\psi}(x)} \, dx$$

for $\phi$ and $\psi$ in the Schwartz space. (This is also sometimes called the Plancherel formula.)

58. In this problem $J_n$ denotes the Bessel function of the first kind and of order $n$. It may defined in various ways, one of which is

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz\cos \theta} \cos(n\theta) \, d\theta$$

Suppose that $f$ is a radially symmetric function in $L^1(\mathbb{R}^2)$, i.e. $f(x) = f(r)$ where $r = |x|$. Show that

$$\hat{f}(y) = 2\pi \int_0^\infty J_0(r|y|) f(r) r \, dr$$

It follows in particular that $\hat{f}$ is also radially symmetric. Using the known identity $\frac{d}{dz}(zJ_1(z)) = zJ_0(z)$ compute the Fourier transform of $\chi_{B(0,R)}$ the indicator function of the ball $B(0, R)$ in $\mathbb{R}^2$. (See problem 11.16 in text for the $\mathbb{R}^3$ case.)

59. For $\alpha \in \mathbb{R}$ let $f_\alpha(x) = \cos \alpha x$.

a) Find the Fourier transform $\hat{f}_\alpha$.

b) Find $\lim_{\alpha \to 0} \hat{f}_\alpha$ and $\lim_{\alpha \to \infty} \hat{f}_\alpha$ in the sense of distributions.

60. Compute the Fourier transform of the Heaviside function $H(x)$ by justifying that

$$\hat{H} = \lim_{n \to \infty} \hat{H}_n$$

where $H_n(x) = H(x)e^{-\frac{x}{n}}$, and then evaluating this limit. (See exercise 11.5 in text.)