MATH 515 Fall 2007 Final Exam

Explain your answers carefully!

1. If $E \subset \mathbb{R}^n$ has finite measure, and f is a bounded, measurable function on E, define the L^p integral average

$$A_p = \left(\frac{1}{m(E)} \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$. Show that if $p < q$ then $A_p \le A_q$.

Solution: By Hölder's inequality, using conjugate exponents $\frac{q}{p}, \frac{q}{q-p}$ we have

$$\int_E |f(x)|^p \, dx \le \left(\int_E |f(x)|^q \, dx \right)^{p/q} \left(\int_E 1 \, dx \right)^{1-p/q}$$

or

$$m(E) A_p^p \le (m(E) A_q^q)^{p/q} m(E)^{1-p/q}$$

Canceling powers of $m(E)$ and taking the p'th root gives $A_p \le A_q$.

2. If X, Y are metric spaces, $f : X \to Y$ is continuous and $K \subset X$ is compact, show that the image $f(K)$ is compact in Y.

Solution: Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$ in Y. Since f is continuous it follows that $f^{-1}(O_\alpha)$ is open in X for every α and so $\{f^{-1}(O_\alpha)\}_{\alpha \in A}$ is an open cover of K. By compactness

$$K \subset \bigcup_{j=1}^{n} f^{-1}(O_{\alpha_j})$$

from which it follows that $\{O_{\alpha_1}, \dots O_{\alpha_n}\}$ is a finite subcover of $f(K)$.

This can also be done by verifying that $f(K)$ is sequentially compact.

3. Let $X = (0,1]$ and $\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.
   (a) Show that $< X, \rho >$ is a metric space.
   (b) Is it a complete metric space?
   (c) Is the metric equivalent to the usual metric $\rho_0(x,y) = |x-y|$?

Solution: Clearly $\rho : X \to [0,\infty)$, $\rho(x,y) = 0$ if and only if $x = y$, $\rho(x,y) = \rho(y,x)$ and

$$\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \le \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = \rho(x,z) + \rho(z,y)$$

so $< X, \rho >$ is a metric space.

$< X, \rho >$ is complete: If $\{x_n\}$ is Cauchy in X, pick N such that $\rho(x_n, x_m) < 1$ for $n, m \ge N$. Then in particular

$$\left|\frac{1}{x_n} - \frac{1}{x_N}\right| \le 1 \qquad n \ge N$$

so that $x_n \ge \delta > 0$ for $n \ge N$ where $\delta = \frac{x_N}{1+x_N}$. Note that

$$|x-y| = xy\rho(x,y) \le \rho(x,y)$$

so that $\{x_n\}$ is also Cauchy in the standard metric. It follows that there exists $x \ge \delta$ such that $x_n \to x$ in the standard metric, Therefore

$$\rho(x_n, x) \le \frac{|x_n - x|}{\delta^2} \qquad n \ge N$$

which implies $x_n \to x$ in $< X, \rho >$.

The metrics are not equivalent: For example

$$\frac{\rho(x,1)}{|x-1|} = \frac{1}{x} \to \infty$$

as $x \to 0$.

4. Let $E = \{f \in C([0,1]) : \int_0^1 f^2(x)\,dx < 1\}$. Show that E is an open, unbounded set in $C([0,1])$.

Solution: Let $f \in E$ and $\delta = 1 - \int_0^1 f^2(x)\,dx > 0$. Let

$$\epsilon = \min\left(1, \frac{\delta}{2||f|| + 1}\right)$$

Then if $g \in C([0,1])$ and $||g - f|| < \epsilon$ we have

$$\int_0^1 g(x)^2\,dx = \int_0^1 (g(x) - f(x))^2 + 2f(x)(g(x) - f(x)) + f(x)^2\,dx$$

$$\le \epsilon^2 + 2\epsilon||f|| + 1 - \delta < 1$$

Thus $B(f, \epsilon) \subset E$, so E is open. (You could also do this by showing E^c is closed.)

If $f_n(x) = \sqrt{n - n^2 x}$ for $0 \le x \le \frac{1}{n}$, $f_n(x) = 0$ otherwise, then $f_n \in C([0,1])$, and $\int_0^1 f_n(x)^2\,dx = \frac{1}{2}$, so $f_n \in E$ for all n, but $||f_n|| = \sqrt{n} \to \infty$ as $n \to \infty$, so E is not bounded.

5. If f is absolutely continuous on $[a,b]$ and $f(x) \neq 0$ on this interval, show that $g(x) = \frac{1}{f(x)}$ is also absolutely continuous.

Solution: Since $f(x) \neq 0$ on $[a,b]$, there must exist $r > 0$ such that $|f(x)| \ge r$ for all $x \in [a,b]$. If $\epsilon > 0$ there must exist $\delta > 0$ such that

$$\sum_{j=1}^n |f(x_j') - f(x_j)| < r^2\epsilon$$

if (x_j, x_j') are disjoint intervals in $[a,b]$ with $\sum_{j=1}^n (x_j' - x_j) < \delta$. Thus we will also then have

$$\sum_{j=1}^n |g(x_j') - g(x_j)| = \sum_{j=1}^n \frac{|f(x_j') - f(x_j)|}{|f(x_j)f(x_j')|}$$

$$\le \frac{1}{r^2}\sum_{j=1}^n |f(x_j') - f(x_j)| < \epsilon$$

for any such collection of intervals. Therefore g is absolutely continuous.

6. Consider the following two properties:

   1) f is continuous almost everywhere.

   2) f is almost everywhere equal to a continuous function.

   Show by example that neither property implies the other.

Solution: The function $f(x) = 0$ for $x < 0$, $f(x) = 1$ for $x > 0$, satisfies 1) but not 2). The function $f(x) = 0$ for $x \in \mathbb{Q}$, $f(x) = 1$ otherwise, satisfies 2) but not 1).

7. Let $f \in L^1(0,1)$ and

$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt \quad 0 < x \leq 1$$

Show that $g \in L^1(0,1)$ and

$$\int_0^1 g(x)\,dx = \int_0^1 f(t)\,dt$$

Solution: We first check that $g \in L^1(0,1)$:

$$\int_0^1 |g(x)|\,dx = \int_0^1 \left| \int_x^1 \frac{f(t)}{t}\,dt \right| dx \leq \int_0^1 \left(\int_x^1 \frac{|f(t)|}{t}\,dt \right) dx$$

$$= \int_0^1 \frac{|f(t)|}{t} \left(\int_0^t dx \right) dt = \int_0^1 |f(t)|\,dt$$

where the exchange of order of integration is justified by Tonelli's Theorem. It follows also that the function

$$F(x,t) = \frac{f(t)}{t} \quad x < t < 1 \qquad F(x,t) = 0 \quad 0 < t < x$$

in in $L^1((0,1)\times(0,1))$. Thus by Fubini's Theorem

$$\int_0^1 g(x)\,dx = \int_0^1\int_x^1 \frac{f(t)}{t}\,dtdx = \int_0^1\int_0^1 F(x,t)\,dtdx$$

$$= \int_0^1\int_0^1 F(x,t)dxdt = \int_0^1 \frac{f(t)}{t}\left(\int_0^t dx\right)dt = \int_0^1 f(t)\,dt$$