1. If $f$ is continuous on $[a, b]$ show that there exists $c \in (a, b)$ such that $\int_a^b f(x) \, dx = f(c)(b - a)$. (Hint: Make use of $F(x) = \int_a^x f(s) \, ds$.)

**Solution:** The hypotheses of the Mean Value Theorem are all satisfied by $F$, so there exists $c \in (a, b)$ such that

$$\int_a^b f(x) \, dx = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a)$$

by the Fundamental Theorem of Calculus.

2. Let $E = [0, 1] \subset \mathbb{R}$.

(a) Show by example that $E$ can be expressed as an intersection of open sets.

(b) Can $E$ be expressed as a union of open sets?

**Solution:** If $E_n = (-\frac{1}{n}, 1 + \frac{1}{n})$ then each $E_n$ is open and $\bigcap_{n=1}^{\infty} E_n = E$. $E$ cannot be expressed as a union of open sets since any union of open sets is open, and $E$ is not open.

3. Let $f(x)$ denote the power series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}(x + 1)^n$.

(a) For which $x$ does the series converge?

(b) Show that $f(x) = x + 1 - \log(x + 2)$ in the interior of the interval of convergence.

**Solution:** The radius of convergence is 1, so the series converges for $|x + 1| < 1$ and diverges for $|x + 1| > 1$. At the endpoint $x = 0$ it is the alternating harmonic series, hence convergent and at $x = -2$ it is the harmonic series so divergent. So the series converges if and only if $x \in (-2, 0]$. 
The series can be differentiated term by term in the interior of this interval, so that
\[ f'(x) = \sum_{n=2}^{\infty} (-1)^n (x + 1)^{n-1} = (x + 1) \sum_{n=0}^{\infty} (- (x + 1))^n \]
The last sum is a geometric series with ratio \(- (x + 1)\) so is equal to
\[ \frac{1}{1 - (- (x + 1))} = \frac{1}{x + 2} \]
Thus we get
\[ f'(x) = \frac{x + 1}{x + 2} \]
Integrate this using \( f(-1) = 0 \) to get the stated formula for \( f \).

4. Let \( f(x) = \frac{x}{e} - \log x \)
   (a) Show that \( f(x) > 0 \) for \( x > e \).
   (b) Use the result of (a) to decide which is larger, \( e^\pi \) or \( \pi^e \)?

**Solution:** We have \( f'(x) = \frac{1}{e} - \frac{1}{x} > 0 \) if \( x > e \). Thus \( f(x) > f(e) = 0 \) if \( x > e \). In particular \( f(\pi) > 0 \) so that \( \frac{\pi}{e} > \log \pi \). Solving and rearranging this inequality gives \( e^\pi > \pi^e \).

5. Find the value of the limit
   \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n + i} \]
   (Suggestion: Find \( a, b, f \) so that the sum \( \sum_{i=1}^{n} \frac{1}{n+i} \) is a Riemann sum for \( \int_{a}^{b} f(x) \, dx \).)

**Solution:** If we let \( f(x) = \frac{1}{x} \) on \([1, 2]\) and choose equally spaced partition points \( x_i = 1 + \frac{i}{n}, i = 0, \ldots, n \) then the above sum is \( \sum_{i=1}^{n} f(x_i) \Delta x_i \) which we can regard as either a Riemann sum or the lower sum for this partition. As \( n \to \infty \) it converges to \( \int_{a}^{b} f(x) \, dx = \log 2 \).
6. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, show that $\sum_{n=1}^{\infty} |a_n|^p$ is convergent for any $p > 1$.

**Solution:** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\lim_{n \to \infty} |a_n| = 0$ and in particular this sequence is bounded, i.e. there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n$. It follows that $|a_n|^p \leq M^{p-1} |a_n|$ so that

$$\sum_{n=1}^{\infty} |a_n|^p \leq M^{p-1} \sum_{n=1}^{\infty} |a_n|$$

and so is convergent.

7. Let $f, g : X \to \mathbb{R}$ and

$$h(x) = (f \vee g)(x) = \max(f(x), g(x))$$

(a) Show that $h(x) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$.

(b) If $f, g$ are continuous, show that $h$ is also continuous.

**Solution:** To show the identity in part a) just consider the two cases $f(x) \geq g(x)$ and $f(x) < g(x)$, e.g. in the first case

$$h(x) = f(x) = \frac{f(x) + g(x) + (f(x) - g(x))}{2} = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

If $f, g$ are both continuous it follows that $f + g$, and then $|f + g|$ are also continuous, and the continuity of $h$ follows.

8. If $p(x)$ is a cubic polynomial with real coefficients, show that $p$ has at least one real root.

**Solution:** If $p(x) = Ax^3 + Bx^2 + Cx + D$ with $A, B, C, D$ real and $A \neq 0$ then you can show that

$$\lim_{x \to +\infty} p(x) = +\infty \quad \lim_{x \to -\infty} = -\infty$$
if $A > 0$ and
\[
\lim_{x \to +\infty} p(x) = -\infty \quad \lim_{x \to -\infty} = +\infty
\]
if $A < 0$. Either way there must exist points $x_1, x_2$ such that $p(x_1) < 0$ and $p(x_2) > 0$. Since $p$ is continuous on any interval, a root must exist by the Intermediate Value Theorem.

9. Let $f_n(x) = \frac{x^n}{1 + x^n}$ for $0 \leq x \leq 2$.

(a) Find the pointwise limit $f(x) = \lim_{n \to \infty} f_n(x)$ on $[0, 2]$.

(b) Does $f_n \to f$ uniformly on $[0, 2]$?

**Solution:** The pointwise limit is

\[
f(x) = \begin{cases} 
0 & 0 \leq x < 1 \\
\frac{1}{2} & x = 1 \\
1 & 1 < x \leq 2
\end{cases}
\]

The convergence cannot be uniform because if it were $f$ would have to be continuous.

10. From the theory of Fourier series it can be shown that

\[
x - x^2 = \sum_{n=1}^{\infty} \frac{8}{((2n-1)\pi)^3} \sin ((2n-1)\pi x) \quad 0 \leq x \leq 1
\]

Using this identity find the value of the infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}
\]

(Suggestion: integrate both sides of the identity from 0 to 1. Justify all steps.)
Solution: The series on the right is uniformly convergent on \([0, 1]\) by the Weierstrass M test, using \(M_n = \frac{8}{((2n - 1)\pi)^3}\). It is therefore correct to integrate term by term to get

\[
\frac{1}{6} = \int_0^1 (x - x^2) \, dx = \sum_{n=1}^{\infty} \frac{8}{((2n - 1)\pi)^3} \int_0^1 \sin ((2n - 1)\pi x) \, dx
\]

The integral on the right is \(\frac{2}{(2n - 1)\pi}\) so we get

\[
\frac{1}{6} = \sum_{n=1}^{\infty} \frac{16}{((2n - 1)\pi)^4}
\]

from which it follows that sum of the requested series is \(\frac{\pi^4}{96}\).