Explain your answers carefully. 10 points each

1. If f is continuous on $[a, b]$ show that there exists $c \in (a, b)$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$. (Hint: Make use of $F(x) = \int_a^x f(s)\,ds$.)

> **Solution:** The hypotheses of the Mean Value Theorem are all satisfied by F, so there exists $c \in (a, b)$ such that
>
> $$\int_a^b f(x)\,dx = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a)$$
>
> by the Fundamental Theorem of Calculus.

2. Let $E = [0, 1] \subset \mathbb{R}$.
   (a) Show by example that E can be expressed as an intersection of open sets.
   (b) Can E be expressed as a union of open sets?

> **Solution:** If $E_n = (-\frac{1}{n}, 1 + \frac{1}{n})$ then each E_n is open and $\cap_{n=1}^{\infty} E_n = E$. E cannot be expressed as a union of open sets since any union of open sets is open, and E is not open.

3. Let $f(x)$ denote the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n}(x+1)^n$.
   (a) For which x does the series converge?
   (b) Show that $f(x) = x + 1 - \log(x+2)$ in the interior of the interval of convergence.

> **Solution:** The radius of convergence is 1, so the series converges for $|x+1| < 1$ and diverges for $|x+1| > 1$. At the endpoint $x = 0$ it is the alternating harmonic series, hence convergent and at $x = -2$ it is the harmonic series so divergent. So the series converges if and only if $x \in (-2, 0]$.

The series can be differentiated term by term in the interior of this interval, so that

$$f'(x) = \sum_{n=2}^{\infty}(-1)^n(x+1)^{n-1} = (x+1)\sum_{n=0}^{\infty}(-(x+1))^n$$

The last sum is a geometric series with ratio $-(x+1)$ so is equal to

$$\frac{1}{1-(-(x+1))} = \frac{1}{x+2}$$

Thus we get

$$f'(x) = \frac{x+1}{x+2}$$

Integrate this using $f(-1) = 0$ to get the stated formula for f.

4. Let $f(x) = \dfrac{x}{e} - \log x$
   (a) Show that $f(x) > 0$ for $x > e$.
   (b) Use the result of (a) to decide which is larger, e^π or π^e?

Solution: We have $f'(x) = \frac{1}{e} - \frac{1}{x} > 0$ if $x > e$. Thus $f(x) > f(e) = 0$ if $x > e$. In particular $f(\pi) > 0$ so that $\frac{\pi}{e} > \log \pi$. Solving and rearranging this inequality gives $e^\pi > \pi^e$.

5. Find the value of the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}$$

(Suggestion: Find a, b, f so that the sum $\sum_{i=1}^{n}\frac{1}{n+i}$ is a Riemann sum for $\int_a^b f(x)\,dx$.)

Solution: If we let $f(x) = \frac{1}{x}$ on $[1, 2]$ and choose equally spaced partition points $x_i = 1 + \frac{i}{n}, i = 0, \ldots n$ then the above sum is $\sum_{i=1}^{n} f(x_i)\Delta x_i$ which we can regard as either a Riemann sum or the lower sum for this partition. As $n \to \infty$ it converges to $\int_a^b f(x)\,dx = \log 2$.

6. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, show that $\sum_{n=1}^{\infty} |a_n|^p$ is convergent for any $p > 1$.

Solution: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\lim_{n\to\infty} |a_n| = 0$ and in particular this sequence is bounded, i.e. there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n. It follows that $|a_n|^p \leq M^{p-1}|a_n|$ so that

$$\sum_{n=1}^{\infty} |a_n|^p \leq M^{p-1} \sum_{n=1}^{\infty} |a_n|$$

and so is convergent.

7. Let $f, g : X \to \mathbb{R}$ and

$$h(x) = (f \vee g)(x) = \max(f(x), g(x))$$

(a) Show that $h(x) = \dfrac{f(x) + g(x) + |f(x) - g(x)|}{2}$.

(b) If f, g are continuous, show that h is also continuous.

Solution: To show the identity in part a) just consider the two cases $f(x) \geq g(x)$ and $f(x) < g(x)$, e.g. in the first case

$$h(x) = f(x) = \frac{f(x) + g(x) + (f(x) - g(x))}{2} = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

If f, g are both continuous it follows that $f + g$, and then $|f + g|$ are also continuous, and the continuity of h follows.

8. If $p(x)$ is a cubic polynomial with real coefficients, show that p has at least one real root.

Solution: If $p(x) = Ax^3 + Bx^2 + Cx + D$ with A, B, C, D real and $A \neq 0$ then you can show that

$$\lim_{x\to+\infty} p(x) = +\infty \qquad \lim_{x\to-\infty} = -\infty$$

if $A > 0$ and

$$\lim_{x\to+\infty} p(x) = -\infty \qquad \lim_{x\to-\infty} = +\infty$$

if $A < 0$. Either way there must exist points x_1, x_2 such that $p(x_1) < 0$ and $p(x_2) > 0$. Since p is continuous on any interval, a root must exist by the Intermediate Value Theorem.

9. Let $f_n(x) = \dfrac{x^n}{1+x^n}$ for $0 \leq x \leq 2$.
   (a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ on $[0,2]$.
   (b) Does $f_n \to f$ uniformly on $[0,2]$?

Solution: The pointwise limit is

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & 1 < x \leq 2 \end{cases}$$

The convergence cannot be uniform because if it were f would have to be continuous.

10. From the theory of Fourier series it can be shown that

$$x - x^2 = \sum_{n=1}^{\infty} \frac{8}{((2n-1)\pi)^3} \sin\left((2n-1)\pi x\right) \qquad 0 \leq x \leq 1$$

Using this identity find the value of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

(Suggestion: integrate both sides of the identity from 0 to 1. Justify all steps.)

Solution: The series on the right is uniformly convergent on $[0,1]$ by the Weierstrass M test, using $M_n = \dfrac{8}{((2n-1)\pi)^3}$. It is therefore correct to integrate term by term to get

$$\frac{1}{6} = \int_0^1 (x - x^2)\,dx = \sum_{n=1}^{\infty} \frac{8}{((2n-1)\pi)^3} \int_0^1 \sin((2n-1)\pi x)$$

The integral on the right is $\dfrac{2}{(2n-1)\pi}$ so we get

$$\frac{1}{6} = \sum_{n=1}^{\infty} \frac{16}{((2n-1)\pi)^4}$$

from which it follows that sum of the requested series is $\dfrac{\pi^4}{96}$.