

Explain your answers carefully.

1. (10 points) Let  $E = \{(-1)^n - \frac{1}{n} : n \in \mathbb{N}\}$ . Find the supremum and infimum of  $E$ .

**Solution:**  $\sup E = 1$  and  $\inf E = -2$ .

2. (10 points) Show that the set of all  $3 \times 5$  matrices with positive rational entries is countable.

**Solution:** If we denote the set of positive rational number by  $\mathbb{Q}^+$  then  $\mathbb{Q}^+$  is countable since it is an infinite subset of a countable set. The set of  $3 \times 5$  matrices with entries in  $\mathbb{Q}^+$  is in an obvious one-to-one correspondence with  $\mathbb{Q}^+ \times \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$  (15 times), and is therefore countable, since any finite Cartesian product of countable sets is countable.

3. (20 points) Let  $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ . Show that  $d$  is a metric on  $\mathbf{X} = (0, 1)$ . Is it a metric on  $\mathbf{X} = [0, 1]$ ?

**Solution:**  $d(x, y)$  is defined for all  $x, y \in (0, 1)$  so we check the metric axioms: (i)  $d(x, y) \geq 0$  clearly and  $d(x, y) = 0$  implies  $\frac{1}{x} = \frac{1}{y}$  which implies  $x = y$ , (ii)  $d(x, y) = |\frac{1}{x} - \frac{1}{y}| = |\frac{1}{y} - \frac{1}{x}| = d(y, x)$ , (iii)  $d(x, y) = |\frac{1}{x} - \frac{1}{y}| \leq |\frac{1}{x} - \frac{1}{z}| + |\frac{1}{z} - \frac{1}{y}| = d(x, z) + d(z, y)$ , by the usual triangle inequality for  $\mathbb{R}$ .  $d$  is *not* a metric on  $[0, 1]$  since  $d(x, y)$  is not defined if  $x$  or  $y$  is 0.

4. (20 points) Let  $E = \mathbb{Q} \cap (0, 1)$ . Is  $E$  open? Is  $E$  closed? Is  $E$  compact? What is  $\overline{E}$ ?

**Solution:** (i)  $E$  is not open since not all of its points are interior points. In fact no point of  $E$  is an interior point (so we say  $E$  has empty interior), since if  $x \in E$  then  $N_r(x)$  always contains irrational numbers for any  $r > 0$ . (ii)  $E$  is not closed since it does not contain all of its limit points, e.g. any irrational number  $x$  is a limit point of  $E$  since  $N_r(x)$  contains points of  $E$  for any  $r > 0$ . (iii)  $E$  is not compact since any compact set is closed. (iv)  $\overline{E} = [0, 1]$ , since for any  $x \in [0, 1]$ ,  $N_r(x) \cap E \neq \emptyset$  for any  $r > 0$ , and for  $x \notin [0, 1]$   $N_r(x) \cap E = \emptyset$  for some  $r > 0$ .

5. (20 points) If  $A, B$  are compact subsets of a metric space  $X$ , show that  $A \cup B, A \cap B$  are also compact.

**Solution:** Let  $\{G_\alpha\}_{\alpha \in A}$  be an open cover of  $A \cup B$ . Then it is also an open cover of  $A, B$  separately. Since  $A$  is compact there exists a finite subcover of  $A$ ,  $G_{\alpha_1}, \dots, G_{\alpha_n}$  and similarly  $G_{\alpha'_1}, \dots, G_{\alpha'_m}$  a finite subcover of  $B$ . It follows that  $G_{\alpha_1}, \dots, G_{\alpha_n}, G_{\alpha'_1}, \dots, G_{\alpha'_m}$  is a finite subcover of  $A \cup B$ , so  $A \cup B$  is compact.

If  $A, B$  are compact then they are closed, so  $A \cap B$  is a closed subset of the compact set  $A$ , hence is compact. (This part could also be done directly from the definition, as in the first part.)

6. (20 points) Let  $A, B \subset \mathbb{R}$  be nonempty and bounded above. Define

$$A + B = \{x \in \mathbb{R} : x = a + b, a \in A, b \in B\}$$

Show that  $\sup(A + B) = \sup A + \sup B$ . (Suggestion: start by showing that  $\sup A + \sup B$  is an upper bound for  $A + B$ .)

**Solution:** If  $a \in A$  and  $b \in B$  then  $a \leq \sup A, b \leq \sup B$  so  $a + b \leq \sup A + \sup B$ , so that  $\sup A + \sup B$  is an upper bound for  $A + B$ . Now suppose that it is not the least upper bound, i.e. there exists  $r < \sup A + \sup B$  which is also an upper bound for  $A + B$ . Pick  $0 < \epsilon < \sup A + \sup B - r$ . There must exist  $a \in A, b \in B$  such that

$$\sup A - \frac{\epsilon}{2} < a \leq \sup A \quad \sup B - \frac{\epsilon}{2} < b \leq \sup B$$

It follows that  $x = a + b \in A + B$  and  $x > \sup A + \sup B - \epsilon > r$ , a contradiction.