MATH 501 FALL 2006 Supplementary Homework 4

Define $L(x) = \int_1^x \frac{1}{t}\,dt$, the natural logarithm function, which we usually denote by $\log x$ or $\ln x$. Derive the following properties using this definition and theorems from class and the text:

1. The domain of L is $(0, \infty)$ and L is continuous on its domain.
2. $L'(x) = \frac{1}{x}$ for $x > 0$.
3. L is strictly monotonically increasing.
4. $L(1) = 0$ and there exists a unique number $e > 1$ such that $L(e) = 1$.
5. $L(xy) = L(x) + L(y)$ for $x, y > 0$. (Hint: $L(xy) = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt$.)
6. $L(x^r) = rL(x)$ for $x > 0$ and $r \in \mathbb{Q}$.
7. $\lim_{x\to\infty} L(x) = +\infty$.
8. $\lim_{x\to 0+} L(x) = -\infty$.
9. The range of L is all of $\mathbb{R}$.
10. L has the inverse function E which satisfies $E'(x) = E(x)$ for all $x \in \mathbb{R}$.
11. $E(x) = e^x$ for $x \in \mathbb{Q}$
12. If we define $x^\alpha = E(\alpha L(x))$ for $\alpha \in \mathbb{R}$, then this agrees with the usual definition for $\alpha \in \mathbb{Q}$, at least for $x > 0$.
13. The derivative of x^α is $\alpha x^{\alpha-1}$ for $x > 0$.
14. $\displaystyle\lim_{x\to\infty} \frac{L(x)}{x^\alpha} = 0$ for any $\alpha > 0$.

For Part 10 you may quote the following theorem: *If f is differentiable on (a, b) and $f' > 0$ on (a, b) then there exists an inverse function $g(x)$, $g(f(x)) = x$ for all $x \in (a, b)$, which is defined and differentiable on the range of f.*