Math 267 Section C1
Exam 2

Name: ____________________

For questions 1 through 3, find the general solution of the given equation

1. $y'' - y = 0$

   Solution: The characteristic equation is $r^2 - 1 = (r-1)(r+1) = 0$
   Therefore the general solution is $y(t) = c_1 e^t + c_2 e^{-t}$.

2. $y^{(3)} - y'' - y' + y = 0$

   Solution: The characteristic equation is $r^3 - r^2 - r + 1 = (r-1)^2(r+1) = 0$
   Therefore the general solution is $y(t) = c_1 e^{-t} + (c_2 + c_3 t)\, e^t$.

3. $y^4 + 2y'' + y = 0$

   Solution: The characteristic equation is $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0$
   Therefore the general solution is $y(t) = (c_1 + c_2 t) \sin t + (c_3 + c_4 t) \cos t$.

4. Find the general solution to $y'' + 3y' + 2y = e^{-t}$ by the method of undetermined coefficients.

Solution: The characteristic equation is $(r+1)(r+2) = 0$. Thus, although we are tempted to use $y(t) = e^{-t}$ as a particular solution, it will not work. We try $y(t) = Ate^{-t}$. We find that this works with $A = 1$.

Therefore the general solution is $y(t) = (c_1 + t)\, e^{-t} + c_2 e^{t}$.

5. Find the general solution to $y''(t) + y = g(t)$, where g is a given function, by the method of variation of parameters. You do not need to evaluate the integrals, just write them out.

Solution: We find that the solutions to the homogeneous equation are $y_1(t) = \sin t$ and $y_2(t) = \cos t$. Then the Wronskian, W, is $-\sin^2 t - \cos^2 t = -1$.

Then our general solution is $y(t) = c_1 \sin t + c_2 \cos t + \int_0^t (\cos s \sin t - \cos t \sin s)\, g(t)\; ds$.

Written more compactly: $y(t) = c_1 \sin t + c_2 \cos t + \int_0^t \sin(t-s) g(t)\; ds$.

6. Solve $y'' + y = 0$ subject to $y(0) = y'(0) = 1$.

Solution: We find our general solution is $y(t) = c_1 \sin t + c_2 \cos t$.

Then $y(0) = c_2$ and $y'(0) = c_1$. Thus $y(t) = \sin t + \cos t$.

For questions 7 through 8, find the Laplace Transform, $Y(s)$ of the solution to the given IVP. You do not need to take the inverse transform.

7. $y'' + y' - 2y = \sin t,\ y(0) = y'(0) = 0$

Solution: Transforming $(s^2 + s - 2)Y(s) = \frac{1}{1+s^2}$. Therefore $Y(s) = \frac{1}{(s^2+s-2)(s^2+1)}$.

8. $y'' + y = H_5(t)e^{t-5},\ y(0) = y'(0) = 0$ where $H_5(t) = \begin{cases} 1 & t \geq 5 \\ 0 & \text{else} \end{cases}$.

Solution: Transforming $(s^2 + 1)Y(s) = \frac{e^{-5s}}{s-1}$. Therefore $Y(s) = \frac{e^{-5s}}{(s-1)(s^2+1)}$.

9. Find the Laplace Transform of $y(t) = e^t \sin t$

Solution: $Y(s) = \frac{1}{(s-1)^2+1}$

10. Find the inverse Laplace Transform of $\dfrac{s}{(s+1)^2 + 4}$

Solution: As, $Y(s) = \frac{s+1}{(s+1)^2+4} - \frac{2}{2((s+1)^2+4)}$.
we see that $y(t) = e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)$