20.4 Acceptance–Rejection Method

The acceptance–rejection method is predicated on the notion of majorization [7]. Suppose we want to sample from a complicated probability density \( f(x) \) that is majorized by a simple probability density \( g(x) \) in the sense that \( f(x) \leq cg(x) = h(x) \) for all \( x \) and some constant \( c > 1 \). If we sample a deviate \( X \) distributed according to \( g(x) \), then we can accept or reject \( X \) as a representative of \( f(x) \). John von Neumann suggested making this decision based on sampling a uniform deviate \( U \) and accepting \( X \) if and only if \( U \leq f(X)/h(X) \). This procedure gives the probability of generating an accepted value in the interval \( (x, x + dx) \) as proportional to

\[
g(x)dx \frac{f(x)}{h(x)} = \frac{1}{c} f(x)dx.
\]

In other words, the density function of the accepted deviates is precisely \( f(x) \). The fraction of sampled deviates accepted is \( 1/c \).

As we have seen in Example 20.2.1, generating exponential deviates is computationally quick. This fact suggests exploiting exponential curves as majorizing functions in the acceptance–rejection method [2]. On a log scale,

![Figure 20.1. Exponential Envelopes for Two Beta Densities](image)

\( \text{Beta}(1, 3) \) and \( \text{Beta}(2, 3) \) distributions.
an exponential curve is a straight line. If a density \( f(x) \) is log-concave, then any line tangent to \( \ln f(x) \) will lie above \( \ln f(x) \). Thus, log-concave densities are ideally suited to acceptance-rejection sampling with piecewise exponential envelopes. Commonly encountered log-concave densities include the normal, the gamma with shape parameter \( \alpha \geq 1 \), the beta with parameters \( \alpha \) and \( \beta \geq 1 \), the exponential power density, and Fisher’s \( z \) density. The reader can easily check log concavity in each of these examples and in the three additional examples mentioned in Problem 5 by showing that \( \frac{d^2}{dx^2} \ln f(x) \leq 0 \) on the support of \( f(x) \).

A strictly log-concave density \( f(x) \) defined on an interval is unimodal. The mode \( m \) of \( f(x) \) may occur at either endpoint or on the interior of the interval. In the former case, we suggest using a single exponential envelope; in the latter case, two exponential envelopes oriented in opposite directions from the mode \( m \). Figure 20.1 depicts the two situations. With different left and right envelopes, the appropriate majorizing function is

\[
h(x) = \begin{cases} 
    c_l \lambda_l e^{-\lambda_l (m-x)} & x < m \\
    c_r \lambda_r e^{-\lambda_r (x-m)} & x \geq m.
\end{cases}
\]

Note that \( h(x) \) has total mass \( c = c_l + c_r \). To minimize the rejection rate and maximize the efficiency of sampling, we minimize the mass constants \( c_l \) and \( c_r \). Geometrically this is accomplished by choosing optimal tangent points \( x_l \) and \( x_r \). The tangency condition for the right envelope amounts to

\[
\begin{align*}
    f(x_r) &= c_r \lambda_r e^{-\lambda_r (x_r-m)} \\
    f'(x_r) &= -c_r \lambda_r^2 e^{-\lambda_r (x_r-m)}.
\end{align*}
\] (2)

These equations allow us to solve for \( \lambda_r \) as \(-f'(x_r)/f(x_r)\) and then for \( c_r \) as

\[
c_r(x_r) = -\frac{f(x_r)^2}{f'(x_r)} e^{-\frac{f'(x_r)}{f(x_r)}(x_r-m)}.
\]

Finding \( x_r \) to minimize \( c_r \) is now a matter of calculus. A similar calculation for the left envelope shows that \( c_l(x_l) = -c_r(x_l) \).
Example 20.4.1 \textit{(Exponential Power Density).} This exponential power density

\[ f(x) = \frac{e^{-|x|^\alpha}}{2\Gamma(1 + \frac{1}{\alpha})}, \quad \alpha \geq 1, \]

has mode \( m = 0 \). For \( x_r \geq 0 \) we have

\[ \lambda_r = \alpha x_r^{\alpha - 1} \]
\[ c_r(x_r) = \frac{e^{(\alpha - 1)x_r^\alpha}}{2\Gamma(1 + \frac{1}{\alpha})\alpha x_r^{\alpha - 1}}. \]

The equation \( \frac{d}{dx} c_r(x) = 0 \) has solution \( -x_l = x_r = \alpha^{-1/\alpha} \). This allows us to calculate the acceptance probability

\[ \frac{1}{2c_r(x_r)} = \Gamma\left(1 + \frac{1}{\alpha}\right)\alpha^{-\frac{1}{\alpha}}e^\frac{1}{\alpha} - 1, \]

which ranges from 1 at \( \alpha = 1 \) (the double or bilateral exponential distribution) to \( e^{-1} = .368 \) as \( \alpha \) tends to \( \infty \). For a normal density \( (\alpha = 2) \), the acceptance probability reduces to \( \sqrt{\pi/2e} \approx .76 \). In practical implementations, the acceptance-rejection method for normal deviates is slightly less efficient than the polar method.