# Proof of Box-Muller Transformation for Normals

This approach is due to Box and Muller . Let us prove that, if $U_1$ and $U_2$ are independent random variates from $\mathcal{U}(0, 1)$, then the variates

$$Z_1 = (-2\ln U_1)^{1/2}\cos 2\pi U_2$$
$$Z_2 = (-2\ln U_1)^{1/2}\sin 2\pi U_2 \qquad (1)$$

are independent standard normal deviates. To see this let us rewrite the system (1) as

$$Z_1 = (2V)^{1/2}\cos 2\pi U$$
$$Z_2 = (2V)^{1/2}\sin 2\pi U, \qquad (2)$$

where $V$ is from exp(1) and $U_2 = U$. It follows from (2) that

$$Z_1^2 + Z_2^2 = 2V \qquad \text{and} \qquad \frac{Z_2}{Z_1} = \tan 2\pi U.$$

The Jacobian of the transformation

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial z_1} & \dfrac{\partial u}{\partial z_2} \\ \dfrac{\partial v}{\partial z_1} & \dfrac{\partial v}{\partial z_2} \end{vmatrix} = \begin{vmatrix} \dfrac{-z_2\cos^2 2\pi u}{2\pi z_1^2} & \dfrac{\cos^2 2\pi u}{2\pi z_1} \\ z_1 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{-z_2}{4\pi v} & \dfrac{z_1}{4\pi v} \\ z_1 & z_2 \end{vmatrix} = -\frac{1}{4\pi v}(z_2^2 + z_1^2) = -\frac{1}{2\pi}$$

and

$$f_{Z_1Z_2}(z_1, z_2) = f_{U,V}(u, v)|J| = \frac{1}{2\pi}\exp\left(-\frac{z_1^2 + z_2^2}{2}\right).$$

The last formula represents the joint p.d.f. of two independent standard normal deviates.