Chapter 5

16. (a) Define \( g \) by

\[
g(x) = \int_a^x f'(t) \, dt.
\]

Then \( g \) is absolutely continuous and \( g' = f' \) a.e. by Theorem 63. It follows that \( h = f - g \) has zero derivative a.e., and, for any \( x > y \), we have

\[
h(x) - h(y) = \int_y^x f'(t) \, dt - f(x) + f(y) \geq 0
\]

by Proposition 51, so \( h \) is increasing.

(b) Let \( \varepsilon > 0 \) be given and set \( \eta = \varepsilon/(b-a) \). Then set

\[
I = \{ [x, x+h] : |f(x) - f(x+h)| < \eta h \}.
\]

Then \( I \) is a Vitali covering of the set on which \( f' = 0 \), so, for a given \( \delta > 0 \), there is a finite collection \( \{ [x_n, x_n+h_n] \} \) of disjoint intervals from \( I \) such that the sums of their lengths is at least \( [a,b] - \delta \). We write them in increasing order (as in the proof of Lemma 62). It follows that

\[
\sum (f(x_n + h_n) - f(x_n)) < \eta \sum h_n \leq \eta (b-a) = \varepsilon.
\]

Setting \( y_0 = a \) and \( y_k = x_{k-1} + h_{k-1} \), we infer that

\[
\sum (f(x_k) - f(y_k)) = f(b) - f(a) - \sum (f(x_n + h_n) - f(x_n)) > f(b) - f(a) - \varepsilon.
\]

(c) For each \( n \), set \( E_n = \{ x \in (a,b) : f'(x) > 1/n \} \). We shall show that \( mE_n = 0 \). Since \( \{ x \in (a,b) : f'(x) \neq 0 \} = \cup E_n \), it will follow that \( f' = 0 \) a.e. as desired.

Let \( \eta > 0 \) be given and let \( n \) be a given positive integer. Then Property (S) gives a finite collection of disjoint intervals \( \{ I_k \} \) with \( I_k = [x_k, y_k] \) for \( x_k \) and \( y_k \) satisfying

\[
a < x_1 < y_1 < x_2 < \cdots < b,
\]

and

\[
\sum \ell(I_k) > (b-a) - \frac{\eta}{2}, \quad \sum f(y_k) - f(x_k) < \frac{\eta}{2n}.
\]

Proposition 51 then gives

\[
\int_{E_n \cap \cup I_k} f' \leq \int_{\cup I_k} f' = \sum \int_{I_k} f' \leq \sum f(y_k) - f(x_k) < \frac{\eta}{2n}.
\]

But

\[
\int_{E_n \cap \cup I_k} f' \geq \frac{1}{n} m(E_n \cap \cup I_k),
\]

and

\[
m(E_n \cap \cup I_k) = m(E_n) - m(E_n \sim \cup I_k) \geq m(E_n) - m([a,b] \sim \cup I_k) \geq m(E_n) - \frac{\eta}{2}.
\]
It follows that
\[ \int_{E_n \cap \cup I_k} f' \geq \frac{1}{n} m(E_n) - \frac{\eta}{2n}. \]
Hence
\[ \frac{1}{n} m(E_n) - \frac{\eta}{2n} < \frac{\eta}{2n}, \]
so \( mE_n < \eta \). Since \( \eta \) is arbitrary, it follows that \( mE_n = 0 \).

(d) Let \( \varepsilon \) and \( \delta \) be given. For each integer \( j \), set
\[ F_j = \sum_{n=1}^{j} f_n, \quad G_j = f - F_j. \]
Now choose \( j \) so that \( G_j(b) < \varepsilon / 2 \). Since \( F_j' = \sum_{n=1}^{j} f'_n \), it follows that \( F_j \) is also singular. Hence, from part (b), there is a finite collection of disjoint intervals \([y_k, x_k]\) with \( \sum y_k - x_k < \delta \) and
\[ \sum F_j(x_k) - F_j(y_k) > F_j(b) - F_j(a) - \varepsilon / 2 \geq f(b) - f(a) - \varepsilon. \]
(Note that \( f(a) \geq F_j(a) \).) Also,
\[ f(x_k) - f(y_k) = F_j(x_k) - F_j(y_k) + G_j(x_k) - G_j(y_k) \geq F_j(x_k) - F_j(y_k) \]
because \( G_j \) is a sum of increasing functions, so it’s also increasing. It follows that
\[ \sum f(x_k) - f(y_k) > f(b) - f(a) - \varepsilon, \]
so \( f \) satisfies Property (S) and hence is singular.

(e) Let \( \langle r_i \rangle \) be an enumeration of the rational numbers, define
\[ f_i(x) = \begin{cases} 0 & \text{if } x < r_i, \\ 2^{-i} & \text{if } x \geq r_i. \end{cases} \]
then \( f_i \) is increasing and \( f'_i = 0 \) everywhere except at \( r_i \), so each \( f_i \) is singular. We then set \( f = \sum f_i \). By part (d), \( f \) is singular. To see that \( f \) is strictly increasing, let \( x < y \). Then there is a rational number \( r_j \) in the interval \((x, y)\), so \( f(y) - f(x) \geq 2^{-j} > 0 \), so \( f \) is strictly increasing.

20. (a) Let \( \varepsilon > 0 \) be given and set \( \delta = \varepsilon / M \) (assuming without loss of generality that \( M > 0 \)).
If \( \{ (x_i, x'_i) \} \) is a collection of nonoverlapping intervals with
\[ \sum |x'_i - x_i| < \delta, \]
then
\[ \sum |f(x'_i) - f(x_i)| \leq \sum M |x'_i - x_i| = M \sum |x'_i - x_i| < M \delta = \varepsilon, \]
so \( f \) is absolutely continuous.

(b) \( \implies \): Since \( f \) satisfies a Lipschitz condition, we have
\[ |f'(x)| = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \leq M \]
because \( (f(x + h) - f(x))/h \leq M \).
\( \iff \): Suppose \( |f'| \leq M \). Then Corollary 64 implies that
\[ f(x) - f(y) = \int_{y}^{x} f'(t) \, dt \]
for any $x > y$. Hence

$$|f(x) - f(y)| \leq \int_y^x |f'| \leq \int_y^x M = M|x - y|.$$  

(c) Suppose $|D^+ f| \leq M$. Define $g$ and $h$ by $g(x) = f(x) + Mx$ and $h(x) = f(x) - Mx$. Then $D^+ g \geq 0$ so $g$ is increasing and $D^+ h \leq 0$ so $h$ is decreasing. If $x > y$, it follows that

$$f(x) - f(y) = g(x) - g(y) - M(x - y) \geq -M(x - y)$$

and

$$f(x) - f(y) = h(x) - h(y) + M(x - y) \leq M(x - y).$$

Combining these two inequalities gives $|f(x) - f(y)| \leq M|x - y|$.

21. (a) Since $O$ is open, we can write it as the union of disjoint open intervals $I_n$. Since $g$ is increasing, $g^{-1}[I_n]$ is an open interval, which we denote by $(a_n, b_n)$. Then $I_n = (g(a_n), g(b_n))$, so

$$m(I_n) = g(b_n) - g(a_n) = \int_{a_n}^{b_n} g'(t) \, dt = \int_{g^{-1}[I_n]} g'(x) \, dx.$$  

It follows that

$$m(O) = \sum m(I_n) = \int_{\bigcup g^{-1}[I_n]} g'(x) \, dx = \int_{g^{-1}[O]} g'(x) \, dx.$$  

(b) For each positive integer $n$, let $H_n = \{ x : g'(x) > 1/n \}$. We shall show that $E_n = g^{-1}(E) \cap H_n$ has measure zero. Since $H = \bigcup H_n$, it follows that $g^{-1}(E) \cap H$ also has measure zero.

Hence, we fix $n$ and let $\varepsilon > 0$ be given. Then there is an open set $O$ such that $E \subset O$ and $mO < \varepsilon/n$. Setting $O_n = g^{-1}[O] \cap H_n$, we have

$$\frac{\varepsilon}{n} > mO = \int_{g^{-1}[O]} g' \geq \int_{O_n} g' \geq \frac{1}{n} m(O_n).$$

It follows that $m(O_n) < \varepsilon$ and hence, because $E_n \subset O_n$, $mE_n < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $mE_n = 0$.

(b) From Proposition 28, there is an $F_\sigma$ set, $E_1$, and a set $E_0$ with measure zero such that $E = E_0 \cup E_1$. Then

$$F = g^{-1}[E_0] \cap H \bigcup g^{-1}[E_1] \cap H.$$  

From part (a), $g^{-1}[E_0] \cap H$ is measurable. In addition $g^{-1}[E_1]$ is a Borel set (by problem 3.26) so it’s measurable, and $H$ is measurable because $g'$ is measurable by virtue of Theorem 52, Theorem 54 and Lemma 60. It follows that $g^{-1}[E] \cap H$ is also measurable.

Part (a) also implies that, for any closed set $C$, we have

$$mC = (d - c) - mO = \int_a^b g' - \int_{g^{-1}[O]} g' = \int_{g^{-1}[C]} g'$$

for $O = [c, d] \sim C$. (This set $O$ may be open or it may be the union of an open set with a one- or two-point set. This second possibility isn’t a problem because points have measure zero.) If $S$ is an arbitrary $F_\sigma$ set, then we have $S = \bigcup C_n$, with each $C_n$ closed and $C_n \subset C_{n+1}$, so

$$mC_n = \int_{g^{-1}[C_n]} g'.$$
Applying Proposition 22 (with $E_n = [c, d] \sim C_n$) shows that $mS = \lim mC_n$ and the monotone convergence theorem (with $f_n = \chi_{g^{-1}[C_n]}$) implies that

$$\lim \int_{g^{-1}[C_n]} g' = \int_{g^{-1}[S]} g',$$

and hence

$$mS = \int_{g^{-1}[S]} g'$$

for any $F_n$, set $S$. Hence

$$mE = mE_1 = \int_{g^{-1}[E_1]} g'.$$

But $g' = 0$ on $g^{-1}[E_1] \sim H$ and $m(g^{-1}[E_0] \cap H) = 0$, so

$$mE = \int_{g^{-1}[E_1] \cap H} g' = \int_F g'.$$

(d) First, $f \circ g$ is the composition of a measurable function with a continuous function, so it's measurable, and we showed in part (b) that $g'$ is measurable.

To prove the integral equality, we first suppose that $f$ is a simple function. Then $f = \sum a_i\chi_{E_i}$ for some finite number of (nonzero) constants $a_i$ and disjoint measurable sets $E_i$. It follows in this case that

$$\int_c^d f(y) \, dy = \sum a_i mE_i = \sum a_i \int_a^b \chi_{E_i}(g(x))g'(x) \, dx = \int_a^b f(g(x))g'(x) \, dx.$$

In general $f$ is the limit of an increasing limit of simple functions $f_n$. It follows that $\langle (f_n \circ g)g' \rangle$ is an increasing sequence of functions, so the Monotone Converge Theorem shows that

$$\int_c^d f(y) \, dy = \int_a^b f(g(x))g'(x) \, dx$$

in this case as well.

25. (a) $\varphi''(t) = p(p-1)(a+tb)^{p-2}b^2$, which is nonnegative if $1 \leq p < \infty$ and nonpositive if $0 < p \leq 1$. Corollary 68 completes the proof.

(b) If $p > 1$, then $\varphi'' > 0$. It follows that $\varphi'$ is strictly increasing. Now let $x < y$ be given, and set

$$\psi(t) = \varphi[t(y + (1-t)x)] - t\varphi(y) - (1-t)\varphi(x)$$

as in the proof of Proposition 67. We see that $\psi'$ is strictly increasing and (as in that proof), we can't have $\psi' > 0$ on the whole interval $(0,1)$. It follows that $\psi$ is initially strictly decreasing, then is zero at one point (the differentiability of $\psi'$ implies that $\psi'$ is continuous), and is positive after that, so it can't be zero anywhere in the interval $(0,1)$. Therefore $\varphi$ is strictly convex. A similar argument applies if $0 < p < 1$.

Chapter 6

2. Set $M_0 = \|f\|_\infty$. then

$$\|f\|_p = \left(\int |f|^p \right)^{1/p} \leq \left(\int M_0^p \right)^{1/p} = M_0.$$

It follows that

$$\lim \|f\|_p \leq M_0.$$
On the other hand, if $M < M_2$, then $m\{x : |f(x)| > M\} = \varepsilon$ is positive, so

$$\int |f|^p \geq M^p \varepsilon,$$

so

$$\|f\|_p \geq M \varepsilon^{1/p}.$$ 

Since $\varepsilon^{1/p} \to 1$ as $p \to \infty$, it follows that

$$\lim\|f\|_p \geq M.$$ 

But $M < M_0$ is arbitrary, so

$$\lim\|f\|_p \geq M_0.$$ 

It follows that $\lim\|f\|_p = M_0$.

8. (a) Set $t = p \ln a$ and $s = q \ln b$, so $a = e^{t/p}$ and $b = e^{s/q}$. Since $e^x$ is a convex function of $x$, it follows that

$$ab = e^{(t/p) + (s/q)} \leq \frac{e^t}{p} + \frac{e^s}{q} = \frac{a}{p} + \frac{b}{q}.$$ 

Since $e^x$ is strictly convex, it follows that equality holds if and only if $t = s$, which means $a^p = b^q$.

(b) Without loss of generality, $\|f\|_p$ and $\|g\|_q$ are positive. Set $F = f/\|f\|_p$ and $G = g/\|g\|_q$. Then Young’s inequality implies that

$$\int |F||G| \leq \frac{1}{p} \int |F|^p + \frac{1}{q} \int |G|^q = \frac{1}{p} + \frac{1}{q} = 1.$$ 

It follows that

$$\int |fg| = \left( \int |F||G| \right) \|f\|_p \|g\|_q \leq \|f\|_p \|g\|_q.$$ 

If equality holds, then we must have $|F|^p = |G|^q$ a.e. and hence $\alpha|f|^p = \beta|g|^q$ a.e.

(c) If $0 < p < 1$, then $q < 0$. Let us set $P = 1/p$ and $Q = 1/(1 - p)$ so that $\frac{1}{P} + \frac{1}{Q} = 1$. Then we take $\alpha = (ab)^{P}$ and $\beta = b^{-p}$. Young’s inequality gives

$$\alpha \beta \leq \frac{\alpha^{P}}{P} + \frac{\beta^{Q}}{Q}.$$ 

Since $q = -pQ$, it follows that

$$a^p \leq pab - \frac{p}{q} b^q,$$

so (after dividing both sides by $p$ and then adding $b^q/q$ to both sides)

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}.$$ 

(d) (We follows the proof of part (b) with the change from Young’s inequality) Without loss of generality, $\|f\|_p$ and $\|g\|_q$ are positive. Set $F = f/\|f\|_p$ and $G = g/\|g\|_q$. Then Young’s inequality implies that

$$\int |F||G| \geq \frac{1}{p} \int |F|^p + \frac{1}{q} \int |G|^q = \frac{1}{p} + \frac{1}{q} = 1.$$ 

It follows that

$$\int |fg| = \left( \int |F||G| \right) \|f\|_p \|g\|_q \geq \|f\|_p \|g\|_q.$$