Chapter 5

6. As a preliminary, we’ll show that
\[
\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B
\]

imply that
\[
\lim_{x \to a} f(x)g(x) = AB
\]

provided \( B > 0 \) using Problem 2.49. By 2.49a and the definition of limit, for any \( \varepsilon > 0 \), there is a number \( \delta \) such that, for all \( x \) with \( 0 < |x - y| < \delta \), we have \( f(x) \leq A + \varepsilon/(2B) \) and
\[
|g(x) - B| < B \min \left\{ 1, \frac{\varepsilon}{(A+1)B} \right\}.
\]

It follows that
\[
f(x)g(x) \leq (A + \varepsilon/2B)B \left( 1 + \min \left\{ 1, \frac{\varepsilon}{(A+1)B} \right\} \right) \leq AB + \varepsilon,
\]

so \( \lim_{x \to a} f(x)g(x) \leq AB \). A similar argument using 2.49b gives \( \lim_{x \to a} f(x)g(x) \geq AB \) and hence \( \lim_{x \to a} f(x)g(x) = AB \).

(a) Note first that there is a positive constant \( h_0 \) such that \( g(\gamma + h) > g(\gamma) \) if \( 0 < h < h_0 \). Hence
\[
D^+ f \circ g(\gamma) = \lim_{h \to 0^+} \frac{f \circ g(\gamma + h) - f \circ g(\gamma)}{h} = \lim_{h \to 0^+} \frac{g(\gamma + h) - g(\gamma)}{h} = D^+ f(c)g'(\gamma).
\]

(b) Now we have \( g(\gamma + h) < g(\gamma) \), so
\[
\lim_{h \to 0^-} \frac{f \circ g(\gamma + h) - f \circ g(\gamma)}{g(\gamma + h) - g(\gamma)} = \lim_{\eta \to 0^+} \frac{f(c - \eta) - f(c)}{-\eta} = \lim_{\eta \to 0^+} \frac{f(c) - f(c - \eta)}{\eta} = D^- f(c).
\]

Now we use Problem 2.49c and d as well to finish.

(c) Note that the finiteness of the derivates implies that \( |f(c + h) - f(c)|/h \) is bounded, say by \( M \), in some neighborhood of \( c \). Then
\[
|f \circ g(\gamma + h) - f \circ g(\gamma)| \leq M |g(\gamma + h) - g(\gamma)|
\]

and, for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |g(\gamma + h) - g(\gamma)| \leq \frac{\varepsilon}{Mh} \) for \( |h| \leq \delta \). It follows that
\[
\left| \frac{f \circ g(\gamma + h) - f \circ g(\gamma)}{h} \right| < \varepsilon
\]

for \( 0 < |h| < \delta \) and hence \( (f \circ g)'(\gamma) = 0 \).
7. (a) Let \( f = h - g \) with \( h \) and \( g \) both increasing. Then
\[
\lim_{x \to c^+} h(x) = \inf_{x > c} h(x), \quad \lim_{x \to c^-} h(x) = \inf_{x < c} h(x)
\]
and similarly for \( g \), so the one-sided limits for \( f \) also exist.

If \( f \) is increasing, set \( M = f(b) - f(a) \). Let \( n \) be an integer and suppose we have \( k \) points \( c_1, \ldots, c_k \) such that \( c_1 < \cdots < c_k \) with \( |f(c_i +) - f(c_i -)| > 1/n \). Then, by monotonicity,
\[
f(c_1 -) \geq f(a), f(c_i -) \geq f(c_{i-1}), f(c_i +) \geq f(c_i).
\]
Hence
\[
k \leq \sum_{i=1}^{k} |f(c_i +) - f(c_i)| \leq f(b) - f(a) = M.
\]
It follows that \( k < M n \), so there are only finitely many points at which the inequality occurs. Since every discontinuity must have this form, the number of discontinuities is countable.

(b) Write the rationals as \( \langle r_i \rangle \) and define \( f_i \) by
\[
f_i(x) = \begin{cases} 
0 & \text{if } x < r_i, \\
1 & \text{if } x \geq r_i.
\end{cases}
\]

Now set \( f(x) = \sum 2^{-i} f_i(x) \). Since \( f \) is a sum of increasing functions, it’s increasing as well. To see where \( f \) is discontinuous, we examine two cases. At a rational number \( r \), we note that \( r = r_j \) for some \( j \). If \( x < r \), then \( f(r) - f(x) \geq 2^{-j} \), so \( f \) is discontinuous at \( r \). If \( x \) is irrational, then, given \( \varepsilon > 0 \), there is a number \( \delta \) such that the interval \([x-\delta, x+\delta]\) contains none of the rational numbers \( r_1, \ldots, r_j \) with \( 2^{-j} < \varepsilon \). It follows that, on this interval, \( f \) can increase by no more than
\[
\sum_{i=j+1}^{\infty} 2^{-i} = 2^{-j} < \varepsilon,
\]
so \( f \) is continuous at \( x \).

10. (a) NO. Set \( y_j = ((j + \frac{1}{2})\pi)^{-1/2} \) and take the \( x_i \)’s in the subdivision to be \( y_j \) for all \( j \)’s less than or equal to some integer \( J \) (so \( n = J + 1 \) and \( x_i = y_{J-i+1} \) for \( 1 \leq i < J \), \( x_0 = -1 \), and \( x_J = 1 \)). Then
\[
\sum_{i=1}^{J+1} |f(x_i) - f(x_{i-1})| \geq \sum_{j=1}^{J} \left| \frac{1}{(j+\frac{1}{2})\pi} + \frac{1}{(j+\frac{3}{2})\pi} \right|.
\]
This is the partial sum of a divergent series (essentially the harmonic series), so we can make it arbitrarily large by choosing \( J \) large enough.

(b) YES. Now take \( y_j = 1/((j + \frac{1}{2})\pi) \). Notice that
\[
g'(x) = \begin{cases} 
2x \sin \left( \frac{1}{x} \right) - \cos x & \text{if } x \neq 0, \\
0 & \text{if } x = 0
\end{cases}
\]
so \( g'(x) \geq -2 \). Hence \( h(x) = g(x) + 2x \) has a nonnegative derivative, so it is increasing. It follows that \( g \) is the difference of two increasing functions: \( h \) and \( k(x) = 2x \).
12. The continuity of \( f \) does not imply that \( f \) is absolutely continuous on \([0, 1]\). Take \( f \) to be the function from problem 10(a). Since this function is not of bounded variation on \([0, 1]\), it isn’t absolutely continuous there. On the other hand, it’s continuously differentiable on \([\varepsilon, 1]\) for any \( \varepsilon \in (0, 1) \) and hence it’s the indefinite integral of its derivative there by the Fundamental Theorem of Calculus.

If \( f \) is continuous and of bounded variation, then \( f' \) is integrable, so \( \int_0^1 f' \) exists. The Lebesgue Convergence Theorem (applied to \( f_n = f \chi_{[1/n, 1]} \)) implies that

\[
\lim_{n \to \infty} \int_{1/n}^1 f' = \int_0^1 f.
\]

Since \( f \) absolutely continuous on \([1/n, 1]\), it follows that \( \int_{1/n}^1 f' = f(1) - f(1/n) \). The continuity of \( f \) then implies that

\[
\lim_{n \to \infty} \int_{1/n}^1 f' = f(1) - f(0).
\]

Hence, for any \( x \in [0, 1] \), we have

\[
f(x) = f(1) - \int_x^1 f' = f(0) + \int_0^x f',
\]

so \( f \) is an indefinite integral of its derivative and hence \( f \) is absolutely continuous.