Chapter 3

13. We will follow the hint.

(a) We just have to show that (ii) $\iff$ (iv). First, we prove $\Rightarrow$: Given $\varepsilon > 0$, let $O$ be an open set such that $E \subset O$ and $m^*(O \sim E) < \varepsilon/2$. Since $O = (O \sim E) \cup E$, and $m^*E < \infty$, it follows that $m^*O < \infty$. From Theorem 11, $O$ is a countable union of disjoint intervals $O = \bigcup I_n$, and $m^*O = \sum \ell(I_n)$, so we can find a positive integer $N$ such that $\sum_{n=1}^N \ell(I_n) > m^*(O) - \varepsilon/2$. Setting $U = \bigcup_{n=1}^N I_n$, we have $U \triangle E = (U \sim E) \cup (E \sim U)$. But $U \sim E \subset O \sim E$, so $m^*(U \sim E) < \varepsilon/2$ and $E \sim U \subset O \sim U = \bigcup_{n=N+1}^\infty I_n$, so $m^*(E \sim U) < \varepsilon/2$. Countable subadditivity implies that $m^*(U \triangle E) < \varepsilon$.

$\Leftarrow$: Given $\varepsilon > 0$, there is a finite union $U = \bigcup_{n=1}^N I_n$ of open intervals such that $m^*(U \triangle E) < \varepsilon/3$. Hence there is a countable collection of open intervals $(J_n)$ such that $U \triangle E \subset \bigcup J_n$ and $\sum \ell(J_n) < 2\varepsilon/3$. Now we define $I_n = J_{n-N}$ for $n > N$ and $O = \bigcup_{n=1}^\infty I_n$. It follows that $E \subset U$ and

$$m^*(O \sim E) \leq m^*(U \triangle E) + m^*(\bigcup J_n) < \varepsilon.$$ 

(c) (i) $\Rightarrow$ (iii): Since $E$ is measurable, so is $\tilde{E}$, so there is an open set $O$ such that $\tilde{E} \subset O$ and $m^*(O \sim \tilde{E}) < \varepsilon$. Now set $F = \tilde{O}$. Then $F \subset E$ and $O \sim \tilde{E} = E \sim F$, so $m^*(E \sim F) < \varepsilon$.

(iii) $\Rightarrow$ (v): Let $F_n$ be the closed set from part (iii) corresponding to $\varepsilon = 1/n$ and set $F = \bigcup F_n$. Then $F \in F_\delta$, $F \subset E$, and $E \sim F \subset E \sim F_n$ for any $n$, so $m^*(E \sim F) < 1/n$ for any positive integer $n$, which implies that $m^*(E \sim F) = 0$.

(v) $\Rightarrow$ (i): From Lemma 29, $E \sim F$ is measurable, so $E = F \cup (E \sim F)$ is also measurable.

14. (a) Let $C_1$ be the set obtained by removing the middle third from $[0,1]$, and define $C_n$ inductively as the set obtained by removing the middle thirds from the intervals of $C_{n-1}$. Since $C_n$ is a finite union of intervals, its outer measure is equal to the sum of the lengths of the intervals, so $m^*(C_n) = \frac{7}{3}m^*(C_{n-1})$. Since $m^*(C_1) = 2/3$, it follows that $m^*(C_n) = (2/3)^n$. In addition, each $C_n$ is closed and hence measurable (by Theorem 27), so we can apply Proposition 22 to conclude that $m^*(C) = \lim_{n \to \infty} (2/3)^n = 0$. 
