Chapter 7

6. (a) Let the ball be $B(x, \delta)$ and let $y \in B(x, \delta)$. Set $\varepsilon = \delta - \rho(x, y)$. If $z \in B(y, \varepsilon)$, then
$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) < \varepsilon,$$
so $B(x, \delta)$ is open.

(b) Write $E$ for $\{x : \rho(x, y) \leq \delta\}$ and let $z \in \tilde{E}$. Then $\varepsilon = \rho(z, y) - \delta > 0$ and, if $w \in B(z, \varepsilon)$, we have
$$\rho(y, w) \geq \rho(y, z) - \rho(z, w) = \varepsilon + \delta - \rho(z, w) > \delta,$$
so $w \in \tilde{E}$. Therefore $\tilde{E}$ is open, so $E$ is closed.

(c) NO. Let $X = \{0, 1\}$ and define $\rho(x, y) = |x - y|$. This is easily seen to be a metric space, but the closure of the ball
$$\{ x : \rho(x, 0) < 1 \}$$
is the set $\{0\}$ while
$$\{ x : \rho(x, 0) \leq 1 \} = X.$$

7. All except $L^\infty$. We go through the spaces in order.

For $\mathbb{R}^n$, the set of points with rational coordinates is dense and countable.

For $C$, we construct a countable dense set of piecewise linear functions. (This avoids the Weierstrass approximation theorem entirely.) For each positive integer $n$, we write $L_n$ for the set of functions $f$ such that, for each integer $m \in \{1, \ldots, n\}$, there is an integer $k(m)$ such that $f(m/n) = k(m)/n$, and $f$ is linear between these points. Each $L_n$ is clearly countable, so $L = \bigcup L_n$ is also countable. To see that $L$ is dense, fix $f \in C$ and $\varepsilon > 0$. Then $f$ is uniformly continuous, so there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ if $|x - y| < \delta$. Choose $n \geq \max\{2/\varepsilon, \delta\}$. Then there is a function $g \in L_n$ such that $|f(x) - g(x)| \leq \varepsilon/2$ if $x$ has the form $m/n$ for some integer $m$. If $x$ is between two such numbers, say $m/n < x < (m + 1)/n$, then
$$g(x) \leq \max\{g(m/n), g(m + 1/n)\} < \max\{f(m/n), f(m + 1/n)\} + \varepsilon/2 < f(x) + \varepsilon,$$
and similarly, $g(x) > f(x) - \varepsilon$, so $\|f - g\| < \varepsilon$ and hence $L$ is dense in $C$.

For $L^\infty$, look at the set of functions
$$\{ \chi_{[0,x]} : x \in [0,1] \}.$$
It’s easy to check that the $L^\infty$ distance between any two of these functions is 1 and that this set is uncountable. Any dense set must contain at least one element within $1/3$ of each $\chi_{[0,x]}$, so any dense set must be uncountable.

For $L^1$, the set $L$ from the discussion of $C$ is dense and countable.
10. (a) Let \( i \) be the identity map. Then \( i \) is continuous if and only if given \( x \in X \) and \( \varepsilon > 0 \), there is a \( \delta_1 > 0 \) such that
\[
\rho(x, y) < \delta_1 \Rightarrow \sigma(x, y) = \sigma(i(x), i(y)) < \varepsilon.
\]
Similarly, \( i^{-1} \) is continuous if and only if given \( x \in X \) and \( \varepsilon > 0 \), there is a \( \delta_2 > 0 \) such that
\[
\sigma(x, y) < \delta_2 \Rightarrow \rho(x, y) = \rho(i^{-1}(x), i^{-1}(y)) < \varepsilon.
\]
The proof is completely by taking \( \delta = \min\{\delta_1, \delta_2\} \).
(b) We first observe that, for any \( n \)-tuple \( (\xi_1, \ldots, \xi_n) \) of nonnegative numbers, we have
\[
\left( \sum \xi_i^2 \right)^{1/2} \leq \left( \left\lfloor \sum \xi_i \right\rfloor \right)^{1/2} = \sum \xi_i,
\]
\[
\sum \xi_i \leq n \max\{\xi_1, \ldots, \xi_n\},
\]
\[
\max\{\xi_1, \ldots, \xi_n\} \leq \left( \sum \xi_i^2 \right)^{1/2}.
\]
Therefore
\[
\rho(x, y) \leq \rho^*(x, y) \leq n \rho^+(x, y) \leq \rho(x, y).
\]
It follows that the implications of part (a) hold with \( \delta = \varepsilon/n \) for any pair of metrics from the list \( \rho, \rho^*, \rho^+ \), and hence these metrics are equivalent.
(c) Take
\[
\sigma(x, y) = \begin{cases} 
1 & \text{if } x \neq y, \\
0 & \text{if } x = y.
\end{cases}
\]
This is not equivalent to \( \rho \) because there is no positive number \( \delta \) such that \( \rho(0, y) < \delta \) implies that \( \sigma(0, y) < 1/2 \). (Here 0 is the origin in \( \mathbb{R}^n \).)

11. (a) First, we note that \( f(s) = s/(1 + s) \) is a strictly increasing function with \( f(0) = 0 \) and \( \lim_{s \to \infty} f(s) = 1 \). Properties i, ii, and iii are then immediate. To check iv, we write
\[
\sigma(x, y) = f(\rho(x, y)) \leq \frac{\rho(x, z)}{1 + \rho(x, z) + \rho(z, y)} + \frac{\rho(z, y)}{1 + \rho(x, z) + \rho(z, y)}
\]
\[
\leq \frac{\rho(x, z)}{1 + \rho(x, z)} + \frac{\rho(z, y)}{1 + \rho(z, y)} = \sigma(x, z) + \sigma(z, y).
\]
To see that \( \sigma \) and \( \rho \) are equivalent, let \( \varepsilon > 0 \) and \( x \in X \) be given. Set \( \delta = \min\{\varepsilon, 1/2\} \).
If \( \rho(x, y) < \delta \), then \( \sigma(x, y) \leq \rho(x, y) \), so \( \sigma(x, y) < \delta \leq \varepsilon \). On the other hand, if \( \sigma(x, y) < \delta \), then \( \sigma(x, y) < \frac{1}{2} \), so \( \rho(x, y) \leq f^{-1}(1/2) = 1 \). Therefore \( \sigma(x, y) \geq \rho(x, y)/2 \) and hence \( \rho(x, y) \leq 2\sigma(x, y) < \varepsilon \).
Finally, since \( f(s) \leq 1 \) for all \( s \), it follows that \( \sigma(x, y) \leq 1 \) for all \( x \) and \( y \) in \( X \).