Extra problem: Give an example of a sequence of measurable functions \( \langle f_n \rangle \), which satisfies the conditions:

1. The sequence converges almost uniformly to a limit function \( f \).
2. The sequence does not converge uniformly to \( f \).
3. There is an integrable function \( g \) such that \( |f_n| \) is less than or equal to \( g \) a.e.

You can choose your measure space any way you want. Explain why your example shows that none of the convergence types: almost everywhere, in measure, in mean, implies uniform convergence.

(Hint: you don’t have to prove that your sequence converges in these senses.)

On the measure space \( X = [0, 1] \) with Lebesgue measure, let \( f_n(x) = x^n \) and \( f(x) = 0 \). To see that \( f_n \) converges almost uniformly to \( f \), let \( \epsilon > 0 \) be given and set \( \eta = \min\{\epsilon, 1\}/2 \) and \( A_\epsilon = [1 - \eta, 1] \). Then \( m A_\epsilon = \eta < \epsilon \) and \( |f_n(x) - f(x)| \leq (1 - \eta)^n \) on \( X \sim A_\epsilon \), so \( f_n \to f \) uniformly on \( X \sim A_\epsilon \). On the other hand, the sequence converges to 1 at \( x = 1 \), so it doesn’t converge to \( f \) at that point and therefore the sequence doesn’t converge uniformly to \( f \). (If you use \( f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } 0 \leq x < 1 \end{cases} \), then a more careful argument shows that the sequence doesn’t converge uniformly to this function either.) Finally, \( g(x) = 1 \) works.

To see why none of the other convergence types imply uniform convergence, we note that implications 5, 6, 12 show that our sequence also converges a.e., in mean, and in measure.

Chapter 7

3. (a) First, we show that \( \rho(x, y) = 0 \) gives an equivalence relation. First, if \( \rho(x, y) = \rho(y, z) = 0 \), then \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) = 0 \) and \( \rho(x, z) \geq 0 \) so \( \rho(x, z) = 0 \), which means the relation is transitive. It is also reflexive and symmetric because of properties ii and iii on page 139.

Now suppose \( x \) and \( x' \) belong to the same equivalence class and \( y \) and \( y' \) belong to the same equivalence class. Then

\[
\rho(x, y) \leq \rho(x, x') + \rho(x', y') + \rho(y', y) = \rho(x', y'),
\]

and a similar argument shows that \( \rho(x', y') \leq \rho(x, y) \). Therefore \( \rho(x, y) \) depends only on the equivalence classes of \( x \) and \( y \). Moreover, if \( x^* \) and \( y^* \) are equivalence classes in \( X^* \), then (using \( x \) and \( y \) to denote elements of these equivalence classes) we have

\[
\rho^*(x^*, y^*) = \rho(x, y) \geq 0.
\]

(This is property i.) Next, \( \rho^*(x^*, y^*) = 0 \) if and only if \( x \) and \( y \) are in the same equivalence class (this is the definition of equivalence class) so

\[
\rho(x^*, y^*) = \text{ if and only if } x^* = y^*.
\]
(This is property ii.) For properties iii and iv, we have
\[ \rho^*(x^*, y^*) = \rho(x, y) = \rho(y, x) = \rho^*(y^*, x^*); \]
\[ \rho^*(x^*, y^*) = \rho(x, y) \leq \rho(x, z) + \rho(z, y) = \rho^*(x^*, z^*) + \rho^*(z^*, y^*). \]

(b) If \( \rho(x, y) < \infty \) and \( \rho(y, z) < \infty \), then \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) < \infty \), so the relation is transitive. Next, \( \rho(x, x) = 0 < \infty \), so it’s reflexive. Finally, if \( \rho(x, y) < \infty \), then \( \rho(y, x) = \rho(x, y) < \infty \), so it’s symmetric.

To show that a part \( P \) is open, let \( x \in P \). Then \( B(x, 1) \subset P \) because every \( y \in B(X, 1) \) satisfies \( \rho(x, y) < 1 < \infty \). To show that \( P \) is closed, note that the complement of \( P \) is the union of all parts different from \( P \) (see page 24), so this complement is open and hence \( P \) is closed.

4. (a) Let \( f \) be a point of closure of \( C \). Then \( f \) is the uniform limit of continuous functions and hence continuous. (In a careful proof, you need to show that the \( L^\infty \) norm of a continuous function is exactly its maximum.)

(b) This time we show that the complement of this set \( E \) is open. Let \( f \in [0, 1] \sim E \). Then \( f \) is nonzero on some subset of \([0, 1/2]\) with positive measure, so there is a positive number \( \varepsilon \) such that the set \( A \subset [0, 1/2] \) on which \( |f| > \varepsilon \) has measure greater than \( \varepsilon \). If \( g \in L^1 \) is a function such that \( \|g - f\|_1 < \varepsilon^2/4 \), then (by Chebycheff’s inequality)
\[ \frac{\varepsilon}{2} m\{x : |g(x) - f(x)| \geq \frac{\varepsilon}{2}\} \leq \frac{\varepsilon^2}{4}, \]
so \( m\{x : |g(x) - f(x)| \geq \varepsilon/2\} \leq \varepsilon/2 \). Therefore \( |g(x) - f(x)| \leq \varepsilon/2 \) on a subset \( B \) of \( A \) with measure at least \( \varepsilon/2 \). It follows that \( |g| > \varepsilon/2 \) on \( B \) and hence \( g \) is in the complement of \( E \), so this complement is open and hence \( E \) is closed.

(c) Call this set \( E \) and let \( x \in E \). Since \( \int x < 1 \), set \( \varepsilon = 1 - \int x \). If \( \|x - y\|_1 < \varepsilon \), then
\[ \int y = \int x + \int (y - x) \leq \int x + \int |y - x| < \int x + \varepsilon < 1. \]