THEOREMS, ETC., FOR MATH 515

Proposition 1 (comment on page 17). If \( A \) is an algebra, then any finite union or finite intersection of sets in \( A \) is also in \( A \).

Proposition 2 (Proposition 1.1). For every \( \mathcal{C} \subset \mathcal{P}(X) \), there is a smallest algebra \( A \) such that \( \mathcal{C} \subset A \).

Proposition 3 (Proposition 1.2). Let \( A \) be an algebra of subsets of \( X \). If \( A_1, \ldots, A_n \) are sets in \( A \), then there are sets \( B_1, \ldots, B_m \) in \( A \) such that \( B_i \cap B_j = \emptyset \) if \( i \neq j \) and

\[
\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j.
\]

Proposition 4 (Proposition 1.3). For every \( \mathcal{C} \subset \mathcal{P}(X) \), there is a smallest \( \sigma \)-algebra \( A \) such that \( \mathcal{C} \subset A \).

Proposition 5 (Corollary 2.6). The intersection of a finite number of open sets is open.

Proposition 6 (Proposition 2.7). An arbitrary union of open sets is open.

Proposition 7 (comment on page 41). \( \emptyset \) and \( \mathbb{R} \) are open.

Proposition 8 (Proposition 2.8). Every open set is a countable union of disjoint open intervals.

Proposition 9 (Proposition 2.9). If \( \mathcal{C} \) is a collection of open sets, then there is a countable subcollection \( \mathcal{C}_0 \) such that

\[
\bigcup_{O \in \mathcal{C}} O = \bigcup_{O \in \mathcal{C}_0} O.
\]

Proposition 10 (Proposition 2.11). For any set \( E \), \( \overline{E} \) is closed.

Proposition 11 (Proposition 2.12). The union of a finite number of closed sets is closed.

Proposition 12 (Proposition 2.13). An arbitrary intersection of closed sets is closed.

Proposition 13 (Proposition 2.14). A set is closed if and only if its complement is open.

Proposition 14 (Proposition 2.10). (a) If \( A \subset B \), then \( \overline{A} \subset \overline{B} \).
(b) \( \overline{A \cup B} = \overline{A} \cup \overline{B} \).

Theorem 15 (Theorem 2.15 (Heine-Borel)). A set \( E \) is closed and bounded if and only if any open cover of \( E \) contains a finite subcover.

Proposition 16 (Proposition 3.1). For any interval \( I \), \( m^* I = \ell(I) \).
Proposition 17 (=Proposition 3.2). If \( \langle A_n \rangle \) is a countable collection of subsets of real numbers, then
\[
m^* \left( \bigcup A_n \right) = \sum m^*(A_n).
\]

Corollary 18 (=Corollary 3.3). If \( E \) is a countable set, then \( m^*E = 0 \)

Corollary 19 (=Corollary 3.4). The set \([0,1]\) is uncountable.

Proposition 20 (=Proposition 3.5). Let \( A \) be a subset of \( \mathbb{R} \).
(a) For any \( \varepsilon > 0 \), there is an open set \( O \) such that \( A \subset O \) and \( m^*O \leq m^* A + \varepsilon \).
(b) There is a \( G \in G_\delta \) such that \( A \subset G \) and \( m^* A = m^* G \).

Proposition 21 (=Proposition 11.1). If \( A \) and \( B \) are measurable sets with \( A \subset B \), then \( m^* A \leq m^* B \).

Proposition 22 (=Proposition 11.2). If \( \langle E_n \rangle \) is a sequence of measurable sets with \( E_{n+1} \subset E_n \) for all \( n \in \mathbb{N} \) and \( \mu E_1 < \infty \), then
\[
\mu \left( \bigcap E_n \right) = \lim_{n \to \infty} \mu E_n.
\]

Proposition 23 (=Proposition 11.3). If \( \langle E_n \rangle \) is a sequence of measurable sets, then
\[
\mu \left( \bigcup E_n \right) \leq \sum_{n=1}^{\infty} \mu E_n.
\]

Proposition 24 (=Proposition 11.4). If \((X, \mathcal{B}, \mu)\) is a measure space, then there is a complete measure space \((X, \mathcal{B}_0, \mu_0)\) such that
(i) \( \mathcal{B} \subset \mathcal{B}_0 \).
(ii) If \( E \in \mathcal{B} \), then \( \mu E = \mu_0 E \).
(iii) \( E \in \mathcal{B}_0 \) if and only if there are sets \( A \in \mathcal{B}_0, B \in \mathcal{B}, \) and \( C \in \mathcal{B} \) such that \( A \subset C, \mu C = 0, \) and \( E = A \cup B \).

Theorem 25 (=Theorem 12.1). If \( \mu^* \) is an outer measure on \( X \), then the class \( \mathcal{B} \) of \( \mu^* \)-measurable sets is a \( \sigma \)-algebra and \( \bar{\mu} \), the restriction of \( \mu^* \) to \( \mathcal{B} \), is a complete measure on \((X, \mathcal{B})\).

Lemma 26 (=Lemma 3.11). The interval \((a, \infty)\) is Lebesgue measurable for any real number \( a \).

Theorem 27 (=Theorem 3.12). Every Borel set is Lebesgue measurable.

Proposition 28 (=Proposition 3.15). Let \( E \) be a subset of \( \mathbb{R} \). Then the following conditions are equivalent.
(i) \( E \) is Lebesgue measurable.
(ii) For every \( \varepsilon > 0 \), there is an open set \( O \) such that \( E \subset O \) and \( m^*(O \sim E) \leq \varepsilon \).
(iii) For every \( \varepsilon > 0 \), there is a closed set \( F \) such that \( F \subset E \) and \( m^*(E \sim F) \leq \varepsilon \).
(iv) There is a \( G_\delta \) set \( G \) such that \( E \subset G \) and \( m^*(G \sim E) = 0 \).
(v) There is an \( F_\sigma \) set \( F \) such that \( F \subset E \) and \( m^*(E \sim F) = 0 \).

If, in addition, \( mE \) is finite, then these conditions are equivalent to
(vi) For every \( \varepsilon > 0 \), there is a finite union of open intervals \( U \) such that \( m^*(U \triangle E) \leq \varepsilon \).
Lemma 29 (=Lemma 3.6, but for general outer measures). If $\mu^*$ is an outer measure on a set $X$ and $E \subset X$ has outer measure zero, then $E$ is $\mu^*$-measurable.

Proposition 30 (=Proposition 11.5). Let $X$ be an arbitrary set, let $\mathcal{B}$ be a $\sigma$-algebra of subsets of $X$, and let $f$ be an extended real-valued function defined on $X$. Then the following statements are equivalent:

(i) $\{x \in X : f(x) < \alpha\} \in \mathcal{B}$ for every real $\alpha$.
(ii) $\{x \in X : f(x) \leq \alpha\} \in \mathcal{B}$ for every real $\alpha$.
(iii) $\{x \in X : f(x) > \alpha\} \in \mathcal{B}$ for every real $\alpha$.
(iv) $\{x \in X : f(x) \geq \alpha\} \in \mathcal{B}$ for every real $\alpha$.

Theorem 31 (=Theorem 11.6). (a) Suppose $f$ and $g$ are measurable and $c \in \mathbb{R}$. Then $f + c$, $cf$, $f + g$, $f \cdot g$, and $f \vee g$ are measurable.
(b) If $(f_n)$ is a sequence of measurable functions, then

$$\sup f_n, \inf f_n, \liminf f_n, \lim f_n$$

are measurable.

Proposition 32 (=Proposition 11.7). Let $(X, \mathcal{B}, \mu)$ be a measure space and let $f$ be a nonnegative $\mu$-measurable function such that $\mu E = 0$ for $E = \{x \in X : f(x) = \infty\}$. Then there is a sequence of simple functions $(\varphi_n)$ such that $\varphi_{n+1} \leq \varphi_n$ and $\varphi_n(x) \to f(x)$ for every $x \notin E$. If $X$ is $\sigma$-finite, the functions $\varphi_n$ can be chosen to vanish outside a set of finite measure.

As mentioned in class, this proposition is also true without the restriction $\mu E = 0$ in which case $\varphi_n(x) \to f(x)$ for every $x \in X$.

Proposition 33 (=Proposition 11.8). Let $(X, \mathcal{B}, \mu)$ be a complete measure space and let $f$ be a $\mu$-measurable function. If $g$ is an extended real-valued function defined on $X$ such that $\mu\{x \in X : f(x) \neq g(x)\} = 0$, then $g$ is also measurable.

Proposition 34 (=Proposition 3.22). Let $f$ be a Lebesgue measurable function on the interval $[a, b]$ and suppose that $mE = 0$ for $E = \{x \in [a, b] : f(x) = \infty\}$. Then for every $\varepsilon > 0$, there are a continuous function $g$ and a step function $h$ and a set $E_\varepsilon$ with $m(E_\varepsilon) < \varepsilon$ such that $|f(x) - g(x)| < \varepsilon$ and $|f(x) - h(x)| < \varepsilon$ for all $x \in [a, b]$. If, in addition, there are real numbers $m$ and $M$ such that $m \leq f(x) \leq M$ for all $x$, then we can choose $g_\varepsilon$ and $h_\varepsilon$ so that $m \leq g_\varepsilon(x) \leq M$ and $m \leq h_\varepsilon(x) \leq M$ for all $x$.

Proposition 35 (=Proposition 3.24, but for arbitrary finite measure spaces). Let $(f_n)$ be a sequence of measurable functions on a finite measure space $(X, \mathcal{B}, \mu)$ and suppose that there is a function $f$ such that $n \to f$ a.e. on $X$. Then, for every pair of positive numbers $\varepsilon$ and $\delta$, there are a positive integer $N$ and a set $A_{\varepsilon, \delta} \subset X$ such that $\mu A_{\varepsilon, \delta} < \eta$ and $|f_n - f| < \varepsilon$ on $A_{\varepsilon, \delta}$ for all $n \geq N$.

Theorem 36 (Egoroff). If $(X, \mathcal{B}, \mu)$, $f_n$, and $f$ are as in Proposition 3.24, then, for any $\eta > 0$, there is a set $A_\eta \subset X$ such that $\mu A_\eta < \eta$ and $f_n \to f$ uniformly on $A_\eta$.

This theorem is stated in the book as Exercise 30 of Chapter 3 only for the special case of Lebesgue measure.
Lemma 37 (=Lemma 4.1). Let \( \varphi \) be a simple function and suppose that \( E_1, \ldots, E_n \) are disjoint measurable sets such that \( \varphi = \sum a_i \chi_{E_i} \) for some real numbers \( a_1, \ldots, a_n \). Then
\[
\int \varphi = \sum_{i=1}^n a_i m_{E_i}.
\]

Proposition 38 (=Proposition 4.2). Let \( \varphi \) and \( \psi \) be simple functions that vanish except on a set of finite measure. Then
\[
\int a\varphi + b\psi = a \int \varphi + b \int \psi
\]
for any real constants \( a \) and \( b \). In addition, if \( \varphi \leq \psi \), then
\[
\int \varphi \leq \int \psi.
\]

Proposition 39 (=Proposition 4.3). Let \( f \) be defined and bounded on a set of finite measure. Then \( f \) is Lebesgue integrable if and only if \( f \) is measurable.

Proposition 40 (=Proposition 4.4). If \( f \) is bounded and Riemann integrable on \([a, b]\), then \( f \) is Lebesgue integrable on \([a, b]\).

Proposition 41 (=Proposition 4.5). Let \( f \) and \( g \) be bounded measurable functions on a set \( E \) of finite measure.

(i) For any real numbers \( a \) and \( b \), we have
\[
\int af + bg = a \int f + b \int g.
\]
(ii) If \( f = g \) a.e., then \( \int f = \int g \).
(iii) If \( f \leq g \) a.e., then \( \int f \leq \int g \). In particular,
\[
\left| \int f \right| \leq \int |f|.
\]
(iv) If \( \alpha \) and \( \beta \) are real constants such that \( \alpha \leq f \leq \beta \) a.e., then
\[
\alpha m_{E} \leq \int_{E} f \leq \beta m_{E}.
\]
(v) If \( A \) and \( B \) are disjoint subsets of \( E \), then
\[
\int_{A \cup B} f = \int_{A} f + \int_{B} f.
\]

Proposition 42 (=Bounded Convergence Theorem, Proposition 4.6). Let \( \langle f_n \rangle \) be a sequence of measurable functions on a set \( E \) of finite measure. Suppose there is a function \( f \) such that \( f_n \to f \) a.e. Suppose also that there is a number \( M \) such that \( |f_n| \leq M \) a.e. for all \( n \). Then \( f_n \to f \).

Theorem 43 (=Fatou’s Lemma, Theorem 11.11). Let \( \langle f_n \rangle \) be a sequence of non-negative measurable functions and suppose there is a function \( f \) such that \( f_n \to f \) a.e. Then
\[
\int f \, d\mu \leq \lim \int f_n \, d\mu.
\]
Theorem 44 (=Monotone Convergence Theorem, Theorem 11.12). Let \( \langle f_n \rangle \) be a sequence of nonnegative measurable functions and suppose there is a function \( f \) such that \( f_n \to f \) a.e. If \( f_n \leq f \) a.e. for all \( n \), then
\[
\int f \, d\mu = \lim \int f_n \, d\mu.
\]

Proposition 45 (=Proposition 11.13). Let \( f \) and \( g \) be nonnegative measurable functions.
(i) If \( a \) and \( b \) are nonnegative constants, then
\[
\int (af + bg) = a \int f + b \int g.
\]
(ii) \( \int f \geq 0 \) with equality if and only if \( f = 0 \) a.e.
(iii) If \( f \geq g \) a.e., then \( \int f \geq \int g \).

Corollary 46 (=Corollary 11.14). Let \( \langle f_n \rangle \) be a sequence of nonnegative measurable functions. Then
\[
\int \sum f_n = \sum \int f_n.
\]

Proposition 47 (=Proposition 11.15). Suppose \( f \) and \( g \) are integrable functions.
(i) Then, for any constants \( a \) and \( b \), we have
\[
\int (af + bg) = a \int f + b \int g.
\]
(ii) If \( h \) is a measurable function such that \( |h| \leq |f| \) a.e., then \( h \) is integrable.
(iii) If \( f \geq g \) a.e., then \( \int f \geq \int g \).

Theorem 48 (=Lebesgue Convergence Theorem, Theorem 11.16). Let \( g \) be a non-negative integrable function, let \( \langle f_n \rangle \) be a sequence of measurable functions, and suppose that \( f = \lim f_n \) exists a.e. If \( |f_n| \leq g \) a.e. for all \( n \), then
\[
\int f = \lim \int f_n.
\]

Theorem 49 (=Theorem 4.17, but for arbitrary measure spaces). Let \( \langle g_n \rangle \) be a sequence of nonnegative measurable functions which converge a.e. to an integrable function \( g \) and let \( \langle f_n \rangle \) be a sequence of measurable functions which converges a.e. to a function \( f \) and suppose that \( |f_n| \leq g_n \) a.e. for all \( n \). If
\[
\int g = \lim \int g_n,
\]
then
\[
\int f = \lim \int f_n.
\]

Lemma 50 (=Vitali’s Covering Lemma, Lemma 5.1). Let \( E \) be a set with finite outer measure and let \( \mathcal{I} \) be a collection of intervals covering \( E \) in the sense of Vitali. Then for every \( \varepsilon > 0 \), there is a finite subcollection \( \{I_1, \ldots, I_N\} \) of \( \mathcal{I} \) such that \( I_i \cap I_j = \emptyset \) if \( i \neq j \) and
\[
m^*(E \sim \bigcup_{n=1}^N I_n) < \varepsilon.
\]
Proposition 51 (=Proposition 5.2). If \( f \) is continuous on \([a, b]\) and if one derivate of \( f \) is nonnegative on \((a, b)\), then \( f \) is increasing on \([a, b]\).

Theorem 52 (=Theorem 5.3). Let \( f \) be increasing on \([a, b]\). Then \( f \) is differentiable a.e., \( f' \) is measurable, and

\[
\int_a^b f'(x) \, dx \leq f(b) - f(a).
\]

Lemma 53 (=Lemma 5.4). If \( f \) is of bounded variation on \([a, b]\), then

\[
T_b^a = P_b^a + N_b^a
\]

and

\[
f(b) - f(a) = P_b^a - N_b^a.
\]

Theorem 54 (=Theorem 5.5). A function \( f \) is of bounded variation on \([a, b]\) if and only if it's the difference of two increasing functions.

Corollary 55 (=Corollary 5.6). If \( f \) is of bounded variation on \([a, b]\), then \( f' \) exists a.e. on \([a, b]\).

Lemma 56 (=Lemma 5.7). Let \( f \) be integrable on \([a, b]\) and define \( F \) by

\[
F(x) = \int_a^x f(t) \, dt.
\]

Then \( F \) is continuous and of bounded variation on \([a, b]\).

Lemma 57 (=Lemma 5.8). If \( f \) is integrable on \([a, b]\) and if

\[
\int_a^x f(t) \, dt = 0
\]

for all \( x \in (a, b) \), then \( f = 0 \) a.e. on \([a, b]\).

Lemma 58 (=Lemma 5.9). If \( f \) is bounded and measurable on \([a, b]\) and if \( F \) satisfies the equation

\[
F(x) = \int_a^x f(t) \, dt + F(a)
\]

for all \( x \in (a, b) \), then \( F' = f \) a.e. on \([a, b]\).

Theorem 59 (=Theorem 5.10). Let \( f \) be integrable on \([a, b]\) and suppose \( F \) is a function such that

\[
F(x) = \int_a^x f(t) \, dt + F(a)
\]

for all \( x \in (a, b) \), then \( F' = f \) a.e. on \([a, b]\).

Lemma 60 (=Lemma 5.11). If \( f \) is absolutely continuous on \([a, b]\), then \( f \) is of bounded variation there.

Corollary 61 (=Corollary 5.12). If \( f \) is absolutely continuous on \([a, b]\), then \( f \) is differentiable a.e. on \([a, b]\).

Lemma 62 (=Lemma 5.13). If \( f \) is absolutely continuous on \([a, b]\) and \( f' = 0 \) a.e. on \([a, b]\), then \( f \) is constant on \([a, b]\).

Theorem 63 (=Theorem 5.14). A function \( F \) is an indefinite integral on \([a, b]\) if and only if \( F \) is absolutely continuous on \([a, b]\).
Corollary 64 (=Corollary 5.15). If $F$ is absolutely continuous on $[a, b]$, then it’s the indefinite integral of its derivative.

Lemma 65 (=Lemma 5.16). If $\varphi$ is convex on an interval $I$, then for any $x$, $x'$, $y$, and $y'$ in $I$ satisfying the inequalities $x \leq x' < y' < y$, we have

$$\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(y') - \varphi(x')}{y' - x'}.$$ 

Proposition 66 (=Proposition 5.17). If $\varphi$ is convex on an interval $(a, b)$, then it’s absolutely continuous on any finite closed subinterval. The left-hand and right-hand derivatives of $f$ exist (and are finite) everywhere in $(a, b)$ and are equal a.e. on $(a, b)$. In addition, the left-hand and right-hand derivatives are increasing, and the left-hand derivative is everywhere less than or equal to the right-hand derivative.

Lemma 67 (=Lemma 5.18). If $\varphi$ is continuous on $[a, b]$ and if one of its derivates is increasing on $(a, b)$, then $\varphi$ is convex.

Corollary 68 (=Corollary 5.19). Suppose $\varphi''$ exists on $(a, b)$. Then $\varphi$ is convex on $(a, b)$ if and only if $\varphi'' \geq 0$ on $(a, b)$.

Proposition 69 (=Proposition 5.20, Jensen’s inequality). Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$ and let $\varphi$ be convex on $\mathbb{R}$. Then

$$\int_X \varphi(f) \, d\mu \geq \varphi\left(\int_X f \, d\mu\right)$$

for any integrable function $f$ such that $\varphi(f)$ is also integrable.

As mentioned in class, for a nonnegative function $f$, this inequality is true without the restriction that $\varphi(f)$ is integrable.

Inequality 70 (=Inequality 6.1, Minkowski’s inequality). For $1 \leq p \leq \infty$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for all $f$ and $g$ in $L^p$. If $1 < p < \infty$, then inequality holds if and only if there are nonnegative constants $\alpha$ and $\beta$, not both zero, such that $\alpha f = \beta g$.

Inequality 71 (=Inequality 6.2). If $f$ and $g$ are in $L^p$ for some $p \in (0, 1)$, then

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p.$$ 

Lemma 72 (=Lemma 6.3). If $p \in [1, \infty)$ and $a$, $b$, and $t$ are nonnegative constants, then

$$(a + tb)^p \geq a^p + ptba^{p-1}.$$ 

Inequality 73 (=Inequality 6.4, Hölder’s inequality). For $1 \leq p \leq \infty$, if $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ (so $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$), then $fg \in L^1$ and

$$\int |fg| \, d\mu \leq \|f\|_p \|g\|_q.$$ 

If $1 < p < \infty$, then equality holds if and only if there are nonnegative constants $\alpha$ and $\beta$, not both zero such that $\alpha |f|^p = \beta |g|^q$.

Proposition 74 (=Proposition 6.5). A normed linear space $X$ is complete if and only if every absolutely summable series is summable.
Theorem 75 (=Theorem 6.6, Riesz-Fischer Theorem). The $L^p$ spaces are complete for $1 \leq p \leq \infty$.

Lemma 76 (=Lemma 6.7). Let $f \in L^p \ (1 \leq p < \infty)$. Then, for any $\varepsilon > 0$, there are a constant $M$ and a bounded, measurable $f_M$ such that $\|f_M\| \leq M$ and $\|f - f_M\| < \varepsilon$.

Proposition 77 (=Proposition 6.8). Let $f \in L^p[a,b]$ for some $p \in [1,\infty)$. Then for every $\varepsilon > 0$, there are a step function $\varphi$ and a continuous function $\psi$ such that $\|f - \varphi\| < \varepsilon$ and $\|f - \psi\| < \varepsilon$.

Proposition 78 (=Proposition 6.9). Let $f \in L^p[a,b]$ for some $p \in [1,\infty)$. Then $T_\Delta(f) \in L^p$ for any subdivision $\Delta$ of $[a,b]$ and $T_\Delta(f)$ converges in $L^p$ mean as the maximum length of the subintervals of $\Delta$ tends to zero.

Proposition 79 (=Proposition 6.11). Let $1 \leq p < \infty$ and let $g \in L^q \ (\frac{1}{p} + \frac{1}{q} = 1)$. Then $F$, defined by

$$F(f) = \int fg,$$

is a bounded linear functional on $L^p$ and $\|F\| = \|g\|_p$.

Lemma 80 (=Lemma 6.12). Let $F$ be a bounded linear functional on $L^p \ (1 \leq p < \infty)$ and suppose there is an integrable function $g$ such that

$$(1) \quad F(f) = \int fg$$

for all $f \in L^\infty \cap L^p$. Then $g \in L^q$, $\|F\| = \|g\|_q$, and (1) is valid for all $f \in L^p$.

Theorem 81 (=Theorem 6.13, Riesz Representation Theorem). Let $F$ be a bounded linear functional on $L^p[a,b] \ (1 \leq p < \infty)$. Then there is a function $g \in L^q$ such that (1) is valid for all $f \in L^p$. In addition, $\|F\| = \|g\|_q$.

Lemma 82 (=Implication 4). Suppose $\{F_n\}$ is a sequence of integrable functions (on an arbitrary measure space) such that $f_n \to f$ in $L^1$ (for some integrable function $f$). Then $f_n \to f$ in measure.

Corollary 83 (=Corollary 6.10). If $f$ is integrable on $[a,b]$, then $T_\Delta(f)$ converges in measure to $f$ as $\delta \to 0$.

Lemma 84 (=Implication 14). Suppose $\{F_n\}$ is a sequence of measurable functions (on an arbitrary measure space) such that $f_n \to f$ a.e. for some function $f$. Suppose also that there is a nonnegative integrable function $g$ such that $|f_n| \leq g$ a.e. for all $n$. Then $f_n \to f$ almost uniformly.

Proposition 85 (=Proposition 7.7). Let $(X, \rho)$ and $(Y, \sigma)$ be metric spaces. Then $f: X \to Y$ is continuous if and only if for every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that, for every $\xi \in X$ with $\rho(x, \xi) < \delta$, we have $\sigma(f(x), f(\xi)) < \varepsilon$.

Proposition 86 (=Proposition 7.6). A metric space $(X, \rho)$ is separable if and only if there is a countable family $\{O_i\}$ of open sets such that every open subset $O$ of $X$ can be written as

$$O = \bigcup_{O_i \in O} O_i.$$
Theorem 87 (=Theorem 7.9). If \((X, \rho)\) is a metric space, then there is a complete metric space \((X^*, \rho^*)\) such that \(X\) is isometrically embedded as a dense subset of \(X^*\). Moreover, if \(X\) is isometrically embedded in some other complete metric space \((Y, \sigma)\), then \(X^*, \rho^*\) is isometric with the closure of the image of \(X\) in \(Y\).

Proposition 88 (=Proposition 7.10). Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces, and let \(f : X \to Y\) be uniformly continuous. If \(\langle x_n \rangle\) is a Cauchy sequence in \(X\), then \(\langle f(x_n) \rangle\) is a Cauchy sequence in \(Y\).

Proposition 89. Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces, and let \(f : X \to Y\) be a function. Then \(f\) is continuous if and only if, whenever \(\langle x_n \rangle\) is a convergent sequence with limit \(x\), then \(f(x_n) \to f(x)\).

Proposition 90 (=Proposition 7.11). Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces, suppose \(Y\) is complete, and let \(f : E \to Y\) be uniformly continuous for some \(E \subset X\). Then there is a unique continuous function \(g : \overline{E} \to Y\) such that \(g|_E = f\). Moreover \(g\) is uniformly continuous.

Proposition 91 (=Proposition 7.13). Every subspace of a separable metric space is separable.

Proposition 92 (=Proposition 7.14). (i) If \(X\) is a metric space and \(A \subset X\) is complete, then \(A\) is closed in \(X\).
(ii) Every closed subspace of a complete metric space is complete

Proposition 93 (=Proposition 7.15). A metric space is compact if and only if every collection of closed subsets \(F\) with the finite intersection property has a non-empty intersection.

Proposition 94 (=Proposition 7.22, first part). A closed subset of a compact space is compact.

Proposition 95 (=Proposition 7.24). The continuous image of a compact space is compact.

Theorem 96 (=Theorem 7.21, Lemma 7.17, Lemma 7.19, and Proposition 7.25). Let \(X\) be a metric space. Then the following conditions are equivalent:
(i) \(X\) is compact.
(ii) \(X\) is complete and totally bounded.
(iii) \(X\) has the Bolzano-Weierstrass property.
(iv) \(X\) is sequentially compact.

Proposition 97 (=Proposition 7.20). Let \(U\) be an open cover of a sequentially compact metric space \(X\). Then there is a positive number \(\varepsilon\) such that, for every \(x \in X\) and every \(\delta \in (0, \varepsilon)\), there is an open set \(O\) in \(U\) such that \(B(x, \delta) \subset O\).

Proposition 98 (=Proposition 7.22, second part). A compact subset of a metric space is closed and bounded.

Corollary 99 (=Corollary 7.23). A subset of the real numbers is compact if and only if it’s closed and bounded.

Proposition 100 (=Proposition 7.18). Let \(f\) be a continuous real-valued function on a compact metric space. Then \(f\) is bounded and it assumes its maximum and minimum values.
Proposition 101 (Proposition 7.26). If \( f \) is a continuous function from a compact metric space \( X \) to a metric space \( Y \), then \( f \) is uniformly continuous.

Proposition 102. A compact metric space is separable.

Theorem 103 (Theorem 7.27, Baire). Let \( \{O_k\} \) be a countable collection of open dense sets in a complete metric space. Then \( \cap O_k \) is dense.

Corollary 104 (Corollary 7.28, Baire Category Theorem). In a complete metric space, no nonempty open subset is of first category.

Proposition 105 (Proposition 7.29). If \( O \) is open and \( F \) is closed in some metric space, then \( \overline{O} \sim O \) and \( \overline{F} \sim F \) are nowhere dense. If \( F \) is closed and of first category in a complete metric space, then \( F \) is nowhere dense.

Proposition 106 (Proposition 7.30). A subset of a complete metric space is residual if and only if it contains a dense \( G_\delta \). Hence a subset of a complete metric space is of first category if and only if it is contained in an \( F_\sigma \) with dense complement.

Proposition 107 (Proposition 7.31). Let \( \{F_n\} \) be a countable collection of closed sets which cover a metric space \( X \). Then \( O = \bigcup F_n \) is a residual open set. If \( X \) is complete, then \( O \) is dense.

Theorem 108 (Theorem 7.32, Uniform Boundedness Principle). Let \( \mathcal{F} \) be a family of real-valued continuous functions on a complete metric space \( X \), and suppose that for any \( x \in X \), there is a number \( M_x \) such that \( |f(x)| \leq M_x \) for all \( f \in \mathcal{F} \). Then there is a nonempty open set \( O \) and a number \( M \) such that \( |f(x)| \leq M \) for all \( f \in \mathcal{F} \) and all \( x \in O \).

Lemma 109 (Lemma 7.37). Let \( \langle f_n \rangle \) be a sequence of functions from a countable set \( D \) into a metric space \( Y \) such that \( \{f_n(x) : n \in \mathbb{N}\} \) has compact closure for each \( x \in D \). Then there is a subsequence \( \langle f_{n_k} \rangle \) which converges at each point of \( D \).

Lemma 110 (Lemma 7.38). Let \( \langle f_n \rangle \) be an equicontinuous sequence of functions from a metric space \( X \) into a complete metric space \( Y \). If there is a dense subset \( D \) of \( X \) such that \( \langle f_n \rangle \) converges at each point of \( D \), then \( \langle f_n \rangle \) converges at each point of \( X \) and the limit function is continuous.

Lemma 111 (Lemma 7.39). Let \( \langle f_n \rangle \) be an equicontinuous sequence of functions from a compact metric space \( K \) into a metric space \( Y \) which converges at each point of \( K \). Then \( \langle f_n \rangle \) converges uniformly.

Theorem 112 (Theorem 7.40, Arzelà-Ascoli). Let \( \mathcal{F} \) be an equicontinuous family of functions from a separable metric space \( X \) into a metric space \( Y \). Let \( \langle f_n \rangle \) be a sequence from \( \mathcal{F} \) such that \( \{f_n(x) : n \in \mathbb{N}\} \) has compact closure for each \( x \in X \). Then there is a subsequence \( \langle f_{n_k} \rangle \) which converges at each point of \( X \) to a continuous function \( f \). Moreover, the convergence is uniform on any compact subset of \( X \).

Corollary 113. Let \( \mathcal{F} \) be an equicontinuous family of real-valued functions on a compact metric space \( X \). Then any bounded subsequence of \( \mathcal{F} \) has a uniformly convergent subsequence.
Corollary 114 (=Corollary 7.41). Let $\mathcal{F}$ be an equicontinuous family of real-valued functions on a separable metric space $X$. Then any sequence from $\mathcal{F}$ which is bounded at each point of a dense subset of $X$ has a subsequence which converges to a continuous function, and the convergence is uniform on each compact subset of $X$.

Theorem 115. In an inner product space,
\[ |(x, y)| \leq \sqrt{(x, x)y^2/2} \]
for all $x$ and $y$.

Theorem 116. In an inner product space,
\[ \|x\| = \sqrt{(x, x)} \]
is a norm.

Theorem 117. In an inner product space,
\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]
for all $x$ and $y$.

Theorem 118. If $B$ is a Banach space and if
\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]
for all $x$ and $y$, then
\[ (x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right) \]
is an inner product on $B$ with $(x, x) = \|x\|^2$.

Lemma 119. Let $M$ be a closed, convex subset of a Hilbert space $H$. Then, for any $x_0 \in H$, there is a unique $y_0 \in M$ such that
\[ \|x_0 - y_0\| = \inf\{\|x_0 - y\| : y \in M\} \]

Theorem 120. Let $M$ be a closed subspace of a Hilbert space $H$. Then, for any $x_0 \in H$, there are unique points $y_0 \in M$ and $z_0 \in M^\perp$ such that $x_0 = y_0 + z_0$.

Theorem 121 (= Proposition 10.28, Riesz representation theorem). If $f$ is a bounded linear functional on a Hilbert space $H$, then there is a unique $z \in H$ such that $f(x) = (x, z)$ for all $x \in H$. Moreover, $\|f\| = \|z\|$.

Proposition 122 (=Proposition 10.27). Let $H$ be a separable Hilbert space.
(a) Every orthonormal system is countable.
(b) If $\langle \varphi_\nu \rangle$ is a complete orthonormal system, then, for every $x \in H$, we have
\[ x = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu \]
with $a_\nu = (x, \varphi_\nu)$. Moreover, $\|x\| = \sum a_\nu^2$.
(c) There is a complete orthonormal system.