A multiplicative Banach-Stone theorem

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Abstract

The Banach-Stone theorem states that any surjective, linear mapping $T$ between spaces of continuous functions that satisfies
\[ \|T(f) - T(g)\| = \|f - g\|, \]
where $\| \cdot \|$ denotes the uniform norm, is a weighted composition operator. We study a multiplicative analogue, and demonstrate that a surjective mapping $T$, not necessarily linear, between algebras of continuous functions with
\[ \|T(f)T(g)\| = \|fg\| \]
must be a composition operator in modulus.

1 Introduction

As stated by Weyl [11, p.144], when studying a mathematical object it is worthwhile to characterize mappings that preserve any structural relations. The collection $C(X)$ of complex-valued continuous functions on a compact Hausdorff space $X$ has many algebraic and topological structures; in particular, when equipped with the uniform norm $\| \cdot \|$, the set $C(X)$ is a normed vector space. It is then advantageous to analyze the mappings that leave this structure undisturbed, and the celebrated Banach-Stone theorem [1, 10] states that any surjective linear isometry between spaces of continuous functions must be a weighted composition operator. More precisely, if $T: C(X) \to C(Y)$ is surjective, linear, and
\[ \|T(f) - T(g)\| = \|f - g\| \]
holds for all $f, g \in C(X)$, then $T(1)$ is unimodular on $Y$ and there exists a homeomorphism $\psi: Y \to X$ such that
\[ T(f) = T(1) \cdot (f \circ \psi). \]
This classic result has since been generalized to mappings between certain linear subspaces, see [2, Chapter 2] for a thorough discussion.

One striking aspect of the Banach-Stone theorem is that it reveals how the vector space structure of $C(X)$ intertwines with the other structures. For example, the mapping $f \mapsto T(1)T(f)$ is an algebra isomorphism, and thus the multiplicative structure of $C(X)$ is preserved, even though the assumptions have no clear connection to point-wise multiplication of functions. A natural question to ask now is if the multiplicative structure of $C(X)$ itself determines any of the other structures.

Recently, there has been much work regarding mappings that leave multiplicative properties invariant under their action. For example, Molnár [9] demonstrated that given a first-countable, compact Hausdorff space $X$ and a surjective, not necessarily linear, mapping $T: C(X) \to C(X)$ that
satisfies $\sigma(T(f)T(g)) = \sigma(fg)$, where $\sigma(\cdot)$ denotes the spectrum, then $T(1)^2 = 1$ and there exists a homeomorphism $\psi: X \to X$ such that $T(f) = T(1) \cdot (f \circ \psi)$, i.e. $T$ is a weighted composition operator. This work has attracted much attention, as its conclusion is reminiscent to that of Banach-Stone, but the linearity of $T$ was not included as an assumption. In light of this, Molnár’s result has inspired a wave of research into similar problems, and it is now considered the inaugural spectral preserver problem. See [3] and the references therein for a general survey on such problems.

Molnár’s assumption of $\sigma(T(f)T(g)) = \sigma(fg)$ implies that
\begin{equation}
\|T(f)T(g)\| = \|fg\|,
\end{equation}
and such mappings have come to be known as norm multiplicative. Analyzing such mappings is the first step in some spectral preserver problems; however, these mappings are of interest in their own right. The condition (1.1) is similar to the assumption of isometry, with subtraction replaced by multiplication, and thus characterizing such mappings can be seen as a multiplicative analogue to the Banach-Stone theorem. It is worth noting that norm multiplicative mappings need not be linear (for example, $T(f) = f$), and any linear $T$ satisfying (1.1) is automatically an isometry, and thus Banach-Stone would apply. Consequently, norm multiplicative mappings are not assumed to be linear.

Mappings $T: A \to B$ between subalgebras $A \subset C(X)$ and $B \subset C(Y)$ that satisfy (1.1) have been studied for a variety of settings, such as Lipschitz algebras [5, Section 3], uniform algebras [7, Theorem 1], and real function algebras [8, Proposition 3.1]. In each case, it is shown that such mappings are composition operators in modulus. This is to say that
\begin{equation}
|T(f)| = |f \circ \psi|,
\end{equation}
where $\psi: \text{Ch}(B) \to \text{Ch}(A)$ is a homeomorphism between the Choquet boundaries, i.e. the points $x \in X$ such that the point-evaluation mapping $\varphi_x: A \to \mathbb{C}$ defined by $\varphi_x(f) = f(x)$ is an extreme point of the unit ball of the dual space of $A$.

As the same conclusion is made, it is natural to consider that there may be a general theorem that envelops the previous work; however the arguments used each situation relies on the fact that $x \in \text{Ch}(A)$ if and only if given an open neighborhood $U$ of $x$, there exists an $f \in A$ such that $\|f\| = |f(x)| = 1$ and $|f| < 1$ on $X \setminus U$. For a general subalgebra $A$, this characterization may fail to be true (see [4] for such an example), and so the usual techniques cannot be used in a wider setting.

In this work, we give a novel approach for characterizing norm multiplicative mappings between general subalgebras that serves as an all encompassing technique. In particular, the conclusion that (1.1) implies (1.2) is strengthened, as the homeomorphism $\psi$ is constructed on the larger Shilov boundary, i.e. points $x \in X$ such that given an open neighborhood $U$ of $x$, there exists an $f$ such that $\|f\| = 1$ and $|f| < 1$ on $X \setminus U$, and this is done using topological divisors of zero, which are functions $f \in A$ such that there exists a normalized sequence $\{h_n\} \subset A$ such that $\|fh_n\| \to 0$. We begin in Section 2 with the basic notation and results that will be needed throughout, and then in Section 3 we prove the following:

Main Theorem. Let $X$ and $Y$ be compact Hausdorff spaces, and let $A \subset C(X)$ and $B \subset C(Y)$ be complex subalgebras that contain the constants and separate points. If $T: A \to B$ is a surjective mapping, not necessarily linear, such that
\[\|T(f)T(g)\| = \|fg\|\]
holds for all \( f, g \in \mathcal{A} \), then there exists a homeomorphism \( \psi: \partial \mathcal{B} \rightarrow \partial \mathcal{A} \) between the Shilov boundaries such that
\[
|T(f)(y)| = |f(\psi(y))|
\]
for all \( f \in \mathcal{A} \) and all \( y \in \partial \mathcal{B} \).

## 2 Notation and Preliminary Results

Throughout this section, \( X \) denotes a compact Hausdorff space, \( C(X) \) denotes the collection of complex-valued continuous functions on \( X \), and \( \mathcal{A} \subset C(X) \) is a complex subalgebra that contains the constants and separates points, i.e. \( \alpha f + \beta g \in \mathcal{A} \) and \( fg \in \mathcal{A} \) for all \( \alpha, \beta \in \mathbb{C} \) and \( f, g \in \mathcal{A} \), \( 1 \in \mathcal{A} \), and given distinct \( x, y \in X \), there exists an \( f \in \mathcal{A} \) such that \( f(x) \neq f(y) \). For \( x \in X \), define
\[
\mathcal{I}_x(\mathcal{A}) = \{ f \in \mathcal{A} : f(x) = 0 \}.
\]
As \( \mathcal{A} \) separates points and \( 1 \in \mathcal{A} \), then \( \mathcal{I}_x(\mathcal{A}) \subset \mathcal{I}_y(\mathcal{A}) \) implies that \( x = y \).

Given \( f \in \mathcal{A} \), the maximizing set of \( f \) is the non-empty, compact set
\[
M(f) = \{ x \in X : |f(x)| = \|f\| \}.
\]
The unit sphere of \( \mathcal{A} \) is the set \( \mathcal{S}(\mathcal{A}) = \{ f \in \mathcal{A} : \|f\| = 1 \} \). A non-empty closed subset \( B \subset X \) is a closed boundary for \( \mathcal{A} \) if \( M(f) \cap B \neq \emptyset \) for all \( f \in \mathcal{S}(\mathcal{A}) \). As \( \mathcal{A} \) separates points, it must be that the intersection of all closed boundaries for \( \mathcal{A} \) is again a closed boundary for \( \mathcal{A} \) [6, Theorem 3.3.2], and this boundary is known as the Shilov boundary, which is denoted by \( \partial \mathcal{A} \). A point \( x \) belongs to \( \partial \mathcal{A} \) if and only if given an open set \( U \) of \( x \), there exists an \( f \in \mathcal{S}(\mathcal{A}) \) such that \( M(f) \subset U \) [6, Corollary 3.3.4].

For \( f_1, \ldots, f_n \in \mathcal{A} \), define
\[
d(f_1, \ldots, f_n) = \inf \left\{ \sum_{j=1}^n \|f_jh\| : h \in \mathcal{S}(\mathcal{A}) \right\}.
\]
A non-empty subset \( \mathcal{J} \subset \mathcal{A} \) consists of joint topological divisors of zero (which will henceforth be abbreviated to JTDZ) if given any finite collection \( f_1, \ldots, f_n \in \mathcal{J} \), then \( d(f_1, \ldots, f_n) = 0 \). Equivalently, \( \mathcal{J} \subset \mathcal{A} \) consists of JTDZ if and only if given \( f_1, \ldots, f_n \in \mathcal{J} \), there exists a sequence \( \{h_n\} \subset \mathcal{S}(\mathcal{A}) \) such that \( \sum_{j=1}^n \|f_jh_n\| \to 0 \).

There is a deep connection between subsets \( \mathcal{J} \) that consist of JTDZ and the Shilov boundary. For any \( x \in \partial \mathcal{A} \), the set \( \mathcal{I}_x(\mathcal{A}) \) consists of JTDZ [6, Theorem 3.4.10], and conversely, any set of JTDZ is contained in \( \mathcal{I}_x(\mathcal{A}) \), for some \( x \in \partial \mathcal{A} \).

**Lemma 2.1.** Let \( \mathcal{J} \subset \mathcal{A} \) consist of JTDZ. Then there exists an \( x \in \partial \mathcal{A} \) such that \( \mathcal{J} \subset \mathcal{I}_x(\mathcal{A}) \).

**Proof.** For each \( f \in \mathcal{J} \), denote the zero set of \( f \) by \( Z(f) = \{ x \in X : f(x) = 0 \} \). We will prove that the family of closed subsets \( \{ Z(f) \cap \partial \mathcal{A} : f \in \mathcal{J} \} \) has the finite intersection property, which yields the desired result.

Indeed, let \( f_1, \ldots, f_n \in \mathcal{J} \) and set \( g = \sum_{j=1}^n |f_j| \). Suppose that \( g \) is non-zero on \( \partial \mathcal{A} \), then there exists an \( \varepsilon > 0 \) such that \( \varepsilon < g(x) \) for all \( x \in \partial \mathcal{A} \). Since \( f_j \in \mathcal{J} \) for all \( 1 \leq j \leq n \), it must be that
\[d(f_1,\ldots, f_n) = 0,\] hence there exists an \(h \in \mathcal{S}(A)\) such that \(\sum_{j=1}^{n} \|f_j h\| < \varepsilon\). Furthermore, as \(\partial A\) is a boundary for \(A\), there exists an \(x_0 \in \partial A\) such \(|h(x_0)| = 1\); however,

\[
\varepsilon < g(x_0) = \sum_{j=1}^{n} |f_j(x_0)| = \sum_{j=1}^{n} |f_j(x_0)h(x_0)| \leq \sum_{j=1}^{n} \|f_j h\| < \varepsilon,\]

which is a contradiction. Therefore, there exists a \(z \in \partial A\) such that \(0 = g(z) = \sum_{j=1}^{n} |f_j(z)|\), and thus \(z \in \bigcap_{j=1}^{n} Z(f_j) \cap \partial A\). \(\square\)

### 3 Proof of Main Theorem

In this section, we prove the following:

**Main Theorem.** Let \(X\) and \(Y\) be compact Hausdorff spaces, and let \(A \subset C(X)\) and \(B \subset C(Y)\) be complex subalgebras that contain the constants and separate points. If \(T: A \to B\) is a surjective mapping, not necessarily linear, such that

\[
\|T(f)T(g)\| = \|fg\| \tag{3.1}
\]

holds for all \(f, g \in A\), then there exists a homeomorphism \(\psi: \partial B \to \partial A\) between the Shilov boundaries such that

\[|T(f)(y)| = |f(\psi(y))|\]

for all \(f \in A\) and all \(y \in \partial B\).

The Main Theorem shall be proven via a sequence of lemmas, and its hypotheses shall be assumed throughout, i.e. \(T: A \to B\) denotes a surjective mapping between complex subalgebras that contain the constants and separates points that satisfies (3.1). Note that (3.1) implies that

\[
\|T(f)\| = \|f\|
\]

for all \(f \in A\), which implies \(\mathcal{S}(A) = T^{-1}[\mathcal{S}(B)]\) and \(T[\mathcal{S}(A)] = \mathcal{S}(B)\).

**Lemma 3.1.** Let \(f_1,\ldots, f_n \in A\). Then \(d(f_1,\ldots, f_n) = d(T(f_1),\ldots, T(f_n))\).

**Proof.** Let \(h \in \mathcal{S}(A)\), then \(T(h) \in \mathcal{S}(B)\). By (3.1),

\[
d(T(f_1),\ldots, T(f_n)) = \inf \left\{ \sum_{j=1}^{n} \|T(f_j)k\| : k \in \mathcal{S}(B) \right\}
\]

\[
\leq \sum_{j=1}^{n} \|T(f_j)T(h)\| = \sum_{j=1}^{n} \|f_j h\|.
\]

As \(h\) was chosen arbitrarily, it must be that \(d(T(f_1),\ldots, T(f_n)) \leq d(f_1,\ldots, f_n)\). A similar argument yields the reverse inequality. \(\square\)

**Lemma 3.2.** Let \(\mathcal{J} \subset A\) and \(\mathcal{D} \subset B\) consist of \(JTDZ\). Then both \(T[\mathcal{J}]\) and \(T^{-1}[\mathcal{D}]\) consist of \(JTDZ\).
Proof. Let \(g_1, \ldots, g_n \in T[J]\), then there exist \(f_1, \ldots, f_n \in J\) such that \(T(f_j) = g_j\) for \(1 \leq j \leq n\). Lemma 3.1 implies that
\[
0 = d(f_1, \ldots, f_n) = d(T(f_1), \ldots, T(f_n)) = d(g_1, \ldots, g_n).
\]
Since \(g_1, \ldots, g_n\) were chosen arbitrarily, it must be that \(T[J]\) consists of JTDZ. The fact that \(T^{-1}[B]\) consists of JTDZ is proven similarly. \(\square\)

**Lemma 3.3.** Let \(y \in \partial B\). Then there exists a unique \(x \in \partial A\) such that \(T^{-1}[I_y(B)] = I_x(A)\).

**Proof.** As \(I_y(B)\) consists of JTDZ, Lemma 3.2 implies that \(T^{-1}[I_y(B)]\) consists of JTDZ. Consequently, Lemma 2.1 guarantees the existence of an \(x \in \partial A\) such that \(T^{-1}[I_y(B)] \subset I_x(A)\).

Now, the surjectivity of \(T\) yields that
\[
I_y(B) = T[T^{-1}[I_y(B)]] \subset T[I_x(A)]
\]
Appealing to Lemmas 2.1 and 3.2 again implies that there exists a \(z \in \partial B\) such that \(I_y(B) \subset T[I_x(A)] \subset I_z(B)\). Since \(B\) separates points and contains 1, it must be that \(y = z\) and thus \(T[I_x(A)] = I_y(B)\). Therefore,
\[
I_x(A) \subset T^{-1}[T[I_x(A)]] = T^{-1}[I_y(B)].
\]
The uniqueness of such an \(x\) follows from the fact that if \(I_w(A) = T^{-1}[I_y(B)] = I_x(A)\), then \(w = x\). \(\square\)

Define the mapping \(\psi: \partial B \to \partial A\) by
\[
T^{-1}[I_y(B)] = I_{\psi(y)}(A).
\]
(3.2)

Note that this is well-defined by the previous lemma.

**Lemma 3.4.** The mapping defined by (3.2) is bijective.

**Proof.** Let \(x \in \partial A\), then there must exist a \(y \in \partial B\) such that \(T[I_x(A)] \subset I_y(B)\). Therefore
\[
I_x(A) \subset T^{-1}[T[I_x(A)]] \subset T^{-1}[I_y(B)] = I_{\psi(y)}(A),
\]
and since \(A\) separates points and 1 \(\in A\), we have that \(\psi(y) = x\). Consequently, \(\psi\) is surjective.

Now, let \(y, z \in \partial B\) be such that \(\psi(y) = \psi(z)\). This implies that
\[
T^{-1}[I_y(B)] = I_{\psi(y)}(A) = I_{\psi(z)}(A) = T^{-1}[I_z(B)],
\]
and the surjectivity of \(T\) yields that
\[
I_y(B) = T[T^{-1}[I_{\psi(y)}(A)]] = T[T^{-1}[I_{\psi(z)}(A)]] = I_z(B)
\]
Therefore \(y = z\), hence \(\psi\) is injective. \(\square\)

We now demonstrate that \(T\) is a composition operator in modulus.

**Lemma 3.5.** Let \(f \in A\) and let \(y \in \partial B\). Then \(|T(f)(y)| = |f(\psi(y))|\).
Proof. Set $\alpha = f(\psi(y))$ and $\beta = T(f)(y)$. As $f - \alpha \in I_{\psi(y)}(A)$, then (3.2) yields that $T(f - \alpha) \in I_y(B)$. Moreover, $T(f) - \beta \in I_y(B)$ and since $I_y(B)$ consists of JTDZ, it must be that $d(T(f - \alpha), T(f) - \beta) = 0$. Thus there exists a sequence $\{k_n\} \subset S(B)$ such that $\|T(f - \alpha)k_n\| + \|T(f) - \beta k_n\| \to 0$, which yields that both $\{\|T(f - \alpha)k_n\|\}$ and $\{\|T(f) - \beta k_n\|\}$ converge to zero.

Let $h_n \in A$ be such that $T(h_n) = k_n$, then $h_n \in S(A)$. Furthermore, (3.1) yields

$$\|fh_n - |\alpha|\| = \|fh_n - |\alpha| h_n\| \leq \|f - \alpha\| h_n = \|T(f - \alpha)k_n\|,$$

and thus $\|fh_n\| \to |\alpha|$. Additionally,

$$\|T(f)k_n - |\beta|\| = \|T(f)k_n - |\beta| k_n\| \leq \|T(f - \beta)k_n\|,$$

which implies that $\|T(f)k_n\| \to |\beta|$.

By (3.1), $\|fh_n\| = \|T(f)T(h_n)\| = \|T(f)k_n\|$ for all $n \in \mathbb{N}$. Therefore, by the uniqueness of limits of sequences for metric spaces, it must be that

$$|T(f)(y)| = |\beta| = |\alpha| = |f(\psi(y))|.$$

We now complete the proof of the Main Theorem.

Lemma 3.6. The mapping defined by (3.2) is a homeomorphism.

Proof. Since $\psi$ is a bijection between a compact space and a Hausdorff space, it is only to show that $\psi$ is continuous. Indeed, let $U \subset \partial A$ be an open set and let $y_0 \in \psi^{-1}[U]$. Since $A$ separates points and contains the complex constant functions, there exist functions $f_1, \ldots, f_n \in A$ such that

$$\psi(y_0) \in \bigcap_{j=1}^n \{x \in \partial A: |f_j(x)| < 1\} \subset U$$

(cf. [6, Proposition 2.2.14]). As $|T(f)(y)| = |f(\psi(y))|$ for all $y \in \partial B$, it follows that

$$y_0 \in \bigcap_{j=1}^n \{y \in \partial B: |T(f_j)(y)| < 1\} \subset \psi^{-1}[U].$$

Therefore $\psi^{-1}[U]$ is open, hence $\psi$ is continuous.

References


