Math 166, Section A1 - Test 03

9 November 2007 Name: ____________________

DIRECTIONS: Answer the following questions or execute the following commands below. You may NOT use a calculator. Remember, you are an attorney and I am a jury of 12 people. You must convince me beyond a reasonable doubt that your answers are correct by showing work and *writing neatly*. **Should you have any questions**, do not hesitate to ask them.

1. Let $a_n = \dfrac{n}{n+1}$.
   (a) Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge? If so, what does it converge to?

   Solution. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$, the sequence converges to 1. □

   (b) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

   Solution. Since $\lim_{n\to\infty} a_n = 1 \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges by the n^{th} term test. □

2. Use the ratio test to show that the series $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ converges.

   Solution. Let $a_n = \dfrac{2^n}{n!}$. Since $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} \dfrac{2^{n+1}n!}{2^n(n+1)!} = \lim_{n\to\infty} \dfrac{2}{n+1} = 0 < 1$, the series $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ converges by the ratio test. □

3. How did we define the term "convergence" for a series? (That is, what is the definition of the phrase "a series converges"?)

   Definition. A series converges if the sequence of its partial sums converges. □

4. Use the ordinary comparison test to show that the series $\sum_{n=1}^{\infty} \dfrac{n}{2n^2-1}$ diverges.

   Solution. Since $\dfrac{n}{2n^2-1} \geq \dfrac{n}{2n^2} = \dfrac{1}{2n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum_{n=1} \dfrac{1}{n}$ diverges ($\sum_{n=1}^{\infty} \dfrac{1}{n}$ is the harmonic series), the series $\sum_{n=1}^{\infty} \dfrac{n}{2n^2-1}$ diverges by the ordinary comparison test. □

5. Does the series $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converge conditionally, converge absolutely, or diverge? Justify your response with an argument that is clear, concise, and easy to follow.

Solution. Since $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is an alternating series and $\lim_{k\to\infty} \frac{1}{k} = 0$, the series converges by the alternating series test.

Since $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (it is the harmonic series), the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally. □

6. List the three types of convergence sets for a power series with center a.

   Solution. The convergence set could be a single point (a), all real numbers, or an interval symmetric about a (plus one or both of the endpoints). □

7. Chose **TWO AND ONLY TWO** of the following problems to complete. Space is provided for you on the following pages; clearly mark which two problems you are attempting.

   (a) Give an example of two divergent series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.

   Solution. There are many examples, but the easiest is to take $a_n = 1$ and $b_n = -1$. Then both $\sum a_n$ and $\sum b_n$ diverge by the n^{th} term test, but the series $\sum a_n + b_n$ converges, as every term in the series is 0, so every term in the sequence of partial sums is 0 (and hence the series converges to 0). □

   (b) How large must n be so that the error obtained by approximating the sum of the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ with the n^{th} partial sum is no more than 0.0002?

   Solution. Recall that the error obtained in using the n^{th} partial sum of the series as an approximation for $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is no larger than $\int_n^{\infty} \frac{1}{x^4}\, dx$. Since

$$\int_n^{\infty} \frac{1}{x^4}\, dx = \lim_{\alpha\to\infty} \int_n^{\alpha} \frac{1}{x^4}\, dx = \lim_{\alpha\to\infty} \Big|_n^{\alpha} - \frac{1}{3x^3} = \lim_{\alpha\to\infty} -\frac{1}{3\alpha^3} + \frac{1}{3n^3} = \frac{1}{3n^3},$$

   we need to pick n large enough to ensure $\frac{1}{3n^3} < 0.0002$. Using basic algebra, we obtain that $n > \sqrt[3]{\frac{5000}{3}}$, so n should be at least 12. □

   (c) Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges at $x = -1$. Can we conclude that the series converges for $x = 6$? Can we conclude that the series converges for $x = 7$?

   Solution. Since $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges for $x = -1$ and the center of the given power series is 3, and because a power series converges absolutely on the interior of its convergence set, the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-3)^n$ is at least 4. Since $6-3=3<4$ the power series converges for $x=6$. Since $7-3=4$, we do not have enough information to determine whether $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges for $x=7$. □

(d) Recall the *Fibonacci sequence* $\{f_n\}$, given by

$$f_1 = f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n.$$

Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} f_n(x-1)^n$.

Solution. Since

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(x-1)^{n+1}}{f_n(x-1)^n} \right| = \varphi|x-1|,$$

the absolute ratio test gives that the series $\sum_{n=1}^{\infty} f_n(x-1)^n$ will converge for all x satisfying $\varphi|x-1|<1$. Then the series converges for $|x-1|<1/\varphi$, so the radius of convergence is $1/\varphi$. □

(e) Give a counterexample to show that the following statement is false: *If $a_n > 0$ for all natural numbers n and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.*

Solution. Set $a_n = 1/n^2$. Then $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges by the p-series test ($p=2>1$), but

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1.$$

□

8. (Bonus points) Which two countries were the main combatants in the Spanish-American war? (You may write your answer below.)

Answer. Spain and the United States □