PATH TO SERIES SUCCESS

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1. Introduction

This document presupposes that you know the working definitions that we have had for the following terms:

- sequence
- series
- convergence (for both sequence and series)
- divergence
- alternating series
- geometric series

2. List of tests for convergence

Below is a list of the tests for convergence/divergence of a series and the order in which to try them.

2.1. The $n^{\mathbf{th}}$ term test.

We always perform this test first because it is the easiest to do. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series. The *n^{th} term test* states the following:

1. If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
2. If $\displaystyle\lim_{n\to\infty} a_n = 0$, then the test is inconclusive.

2.2. The alternating series test.

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be an alternating series. The *alternating series test* states the following:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Please take note how this differs from the n^{th} term test and realize that the two are different. The alternating series test *only* works on an alternating series, so you must establish that $\displaystyle\sum_{n=1}^{\infty} a_n$ is alternating before applying the test.

Date: October 24, 2007.

2.3. **The geometric series test.** Let $\sum_{n=1}^{\infty} a_n$ be a geometric series with ratio r. The *geometric series test* states the following:

1. If $|r| < 1$, then the series converges.
2. If $|r| \geq 1$, then the series diverges.

Again, note that one must first establish that the series $\sum_{n=1}^{\infty} a_n$ is geometric before applying this test.

2.4. **The p-series test.** Let $\sum_{n=1}^{\infty} \frac{1}{n^p}$ be a series, where p is a real number. The *p-series test* states the following:

1. If $p > 1$, then the series converges.
2. If $p \leq 1$, then the series diverges.

2.5. **The ratio test.** Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. The *ratio test* states the following: If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho,$$

then

1. If $\rho < 1$, the series converges;
2. if $\rho > 1$, the series diverges;
3. if $\rho = 1$, then the test is inconclusive.

This test is most effective on series whose sequences of terms have factorials and exponential functions all over the place.

2.6. **The integral test.** Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. The *integral test* states the following: If

$$\int_1^{\infty} a_x \, dx < \infty,$$

then the series $\sum_{n=1}^{\infty} a_n$ converges.

Recall that the meaning of a_x is just "put x where you see n".

2.7. **The limit comparison test.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both be series of positive terms. The *limit comparison test* states the following: If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

and

$$0 < L < \infty,$$

then

$$\text{the series } \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ converge or diverge together.}$$

Furthermore, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

and

$$\sum_{n=1}^{\infty} b_n \text{ converges,}$$

then

$$\sum_{n=1}^{\infty} a_n \text{ also converges.}$$

Note that the to use the limit comparison test, one must already know whether one of the series $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} b_n$ converges (or diverges).

2.8. **The ordinary comparison test.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both be series of positive terms, and suppose further that $a_n \leq b_n$ "eventually" (the meaning of this was discussed in class). The *ordinary comparison test* states the following:

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

This test is probably the easiest to understand in concept, but the hardest to use, since it requires insight to find the proper series with which to compare the given series.

2.9. **The definition of convergence.** We *defined* the phrase "the series $\sum_{n=1}^{\infty} a_n$ converges" to mean "the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ converges". It is always legitimate to fall back on this definition but is hard (in practice) to use. This is why we have all of the other tests for convergence.

3. Examples

Below is a list of sample problems and their solutions using each of the different tests. Pay close attention to the way the solutions are worded so that you can mimic this language when you try on your own.

You are expected to be able to write solutions of the caliber you see here. The solution is everything between the word *Solution* and the box (□). The comments that follow are to help you in understanding the solution.

Essentially, each example solution below contains the bare minimum of work you need to show concerning computations; i.e., each one of these solutions is streamlined and perfect in the eyes of the grader.

3.1. Using the n^{th} term test.

Example 3.1.1. *Does* $\sum_{n=1}^{\infty} \frac{n^2+4}{3n^2-2n+1}$ *converge or diverge?*

Solution. Since

$$\lim_{n\to\infty} \frac{n^2+4}{3n^2-2n+1} = 1/3 \neq 0,$$

the series $\sum_{n=1}^{\infty} \frac{n^2+4}{3n^2-2n+1}$ *diverges* by the n^{th} term test. ☐

Note that we do not have to use l'Hôpital's rule in computing the limit in example 3.1.1. The limit above is something a Calculus I student could do.

3.2. Using the alternating series test.

Example 3.2.1. *Does* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converge or diverge?*

Solution. Note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is an alternating series. Since

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0,$$

the series converges by the alternating series test. ☐

Note how the solution to example 3.2.1 absolutely depends on the series being an alternating series. Do not even attempt using the alternating series test on a series that is not alternating. In other words, you *must* mention that series is alternating before attempting the test. It is enough to simply say that the series is alternating, you do not have to "prove" that.

However, if the series is not alternating, and you claim that it is just so you can get by, you will be penalized for doing so.

3.3. Using the geometric series test.

Example 3.3.1. *Does* $\sum_{n=0}^{\infty} \frac{\pi}{2^n}$ *converge or diverge? If it converges, to what number does it converge?*

Solution. Note that $\sum_{n=0}^{\infty} \frac{\pi}{2^n}$ is a geometric series with ratio $\frac{1}{2}$. Since $\left|\frac{1}{2}\right| = \frac{1}{2} < 1$, the series $\sum_{n=0}^{\infty} \frac{\pi}{2^n}$ converges.

Since the first term in the series is π, the series converges to

$$\frac{\pi}{1-1/2} = 2\pi.$$

☐

Similar to example 3.2.1, it is absolutely necessary that we mention that the series in example 3.3.1 is a geometric series. Again, this amounts to no more than just saying that it is, but we have to be sure. If you are not sure, you may want to write some scratch work to show that the series is geometric.

3.4. **Using the p-series test.**

Example 3.4.1. *Does* $\sum_{k=4}^{\infty} \frac{1}{k^{1.0001}}$ *converge or diverge?*

Solution. This series is a p-series with $p = 1.0001$. Since $1.0001 > 1$, the series converges. ☐

3.5. **Using the ratio test.**

Example 3.5.1. *Does* $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *converge or diverge?*

Solution. Let $a_n = \frac{2^n}{n!}$. Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1} n!}{2^n (n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0,$$

the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the ratio test. ☐

3.6. **Using the integral test.**

Example 3.6.1. *Does* $\sum_{n=1}^{\infty} ne^{-n^2}$ *converge or diverge?*

Solution. Since

$$\begin{aligned} \int_1^{\infty} xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_1^{b} xe^{-x^2}\,dx \\ &= \lim_{b\to\infty} \Big[_1^{\infty} -\frac{1}{2}e^{-x^2} \\ &= \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1} \\ &= \frac{1}{2e} \\ &< \infty, \end{aligned}$$

the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges by the integral test. ☐

Take a second to note that the only thing in the integral test that matters is that the integral is finite.

3.7. Using the limit comparison test.

Example 3.7.1. *Does* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+19n}}$ *converge or diverge?*

Solution. For large n, note that $\frac{1}{\sqrt{n^2+19n}}$ behaves like $1/n$. Note that

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+19n}}}{1/n} &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+19n}} \\ &= 1. \end{aligned}$$

Since $0 < 1 < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+19n}}$ diverges by the limit comparison test. ☐

3.8. Using the ordinary comparison test.

We'll have two examples here.

Example 3.8.1. *Does* $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ *converge or diverge?*

Solution. Since

$$\frac{1}{n^2} \geq \frac{1}{n^2+4},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges by the ordinary comparison test. ☐

Example 3.8.2. *Does* $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$ *converge or diverge?*

Solution. Since

$$\frac{1}{n} \leq \frac{n}{n^2-1}$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test, $\sum_{n=1}^{\infty} \frac{n}{n^2-1}$ diverges by the ordinary comparison test. ☐

Both examples 3.8.1 and 3.8.2 make seemingly unsubstantiated claims (the inequalities). Again, this is fine *if* the inequalities are true. If you make such a claim and it is wrong, then you will be penalized. It never hurts to show a bit more work to convince yourself that your inequalities are true. Don't just write them down for no reason at all.

Again, the solutions here are *streamlined.*

3.9. Using the sequence of partial sums.

Example 3.9.1. *Does* $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ *converge or diverge?*

Solution. Let S_k denote the k^{th} partial sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$. Then

$$\begin{aligned} S_k &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right) \\ &= 1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cdots+\cancel{\frac{1}{k}}-\frac{1}{k+1} \\ &= 1-\frac{1}{k+1} \end{aligned}$$

Since $\lim_{k\to\infty} S_k = \lim_{k\to\infty} 1-\frac{1}{k+1} = 1$, the sequence of partial sums of $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges. Therefore, the series converges. ☐

The most important thing to note here is that we had to switch to using the sequence of partial sums in example 3.9.1, since we are not allowed to arbitrarily group the terms in a series unless we already know it converges.

4. Useful Maclaurin Series

What follows is a list of useful Maclaurin series. It would behoove one to memorize them, although it is not necessary.

4.1. Maclaurin series for $\frac{1}{1-x}$.

4.1.1. *First few terms.* $1+x+x^2+x^3+x^4+\cdots$

4.1.2. *Sigma notation.* $\sum_{n=0}^{\infty} x^n$

4.1.3. *Convergence set.* $-1<x<1$

4.2. Maclaurin series for $\ln(1+x)$.

4.2.1. *First few terms.* $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

4.2.2. *Sigma notation.* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

4.2.3. *Convergence set.* $-1<x\leq 1$

4.3. Maclaurin series for $\arctan(x)$.

4.3.1. *First few terms.* $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

4.3.2. *Sigma notation.* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1}$

4.3.3. *Convergence set.* $-1 \leq x \leq 1$

4.4. Maclaurin series for e^x.

4.4.1. *First few terms.* $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

4.4.2. *Sigma notation.* $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4.4.3. *Convergence set.* All real numbers.

4.5. Maclaurin series for $\sin x$.

4.5.1. *First few terms.* $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

4.5.2. *Sigma notation.* $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

4.5.3. *Convergence set.* All real numbers.

4.6. Maclaurin series for $\cos x$.

4.6.1. *First few terms.* $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

4.6.2. *Sigma notation.* $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

4.6.3. *Convergence set.* All real numbers.

4.7. Maclaurin series for $(1+x)^p$ where p is a real number.

4.7.1. *First few terms.* $1 + \binom{p}{1} x + \binom{p}{2} x^2 + \binom{p}{3} x^3 + \binom{p}{4} x^4 + \cdots$

4.7.2. *Sigma notation.* $\displaystyle\sum_{k=0}^{\infty} \binom{p}{k} x^k$

4.7.3. *Convergence set.* $-1 < x < 1$