

Math 515  
Real Analysis  
Problem Set 3

Due date: September 30, 2005

Consultation with other individuals is not permitted for this problem set.

Each graded problem is worth 10 points.

1. Show that the Lebesgue Dominated Convergence Theorem is false for Riemann Integrals.
2. Use Egoroff's theorem to prove the Lebesgue Dominated Convergence Theorem for measurable functions (which need not be bounded).
3. Show that  $f(x) \equiv x^{-a}$  for  $0 < x \leq 1$  is integrable in the sense of Lebesgue if and only if  $a < 1$  and evaluate the integral in the latter case.
4. Let  $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$  and  $h(x) = f'(x)$  on  $0 < x \leq 1$ . (Here we mean the usual calculus notion of a derivative.) Show that  $h(x)$  is not Lebesgue integrable but that the improper Riemann integral of  $h(x)$  in the sense that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 h(x) dx$$

exists.

5. Suppose  $f \geq 0, g \geq 0$ . If

$$f_n(x) \equiv \begin{cases} n & \text{if } f(x) > n \\ f(x) & \text{if } f(x) \leq n. \end{cases}$$

Show that

$$(f + g)_n \leq f_n + g_n \leq (f + g)_{2n}.$$

6. Let  $C$  be the Cantor set in  $[0, 1]$ . Let

$$f(x) = \begin{cases} 0 & \text{if } x \in C \\ p & \text{if } x \in \text{a complementary interval of length } 3^{-p}. \end{cases}$$

Prove that

$$\int_0^1 f(x) dm(x) = 3.$$

7. Show that  $f$  is absolutely integrable on a set  $E$  if and only if

$$\sum_{n=-\infty}^{+\infty} |n| m(E \cap f^{-1}([n-1, n])) < +\infty.$$