# Reaction-Diffusion Equations and Inverse problems

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## • Content

1. Inverse problems
2. Reaction-Diffusion equations
3. Fisher equation-constant diffusion coefficient

(Based on [6])

## 1. Inverse problems

- Direct problem

$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

Then the solution is

$$\begin{cases} x = 1 \\ y = 1 \end{cases}$$

• Inverse problem

$$\begin{cases} ax + y = 2 \\ x - y = 0 \\ 2x + y = 3 \end{cases}$$

Suppose above system of equation has unique solution, then

$$a = 1$$

► The inverse problems is to seek the coefficients (properties) of equations from the additional conditions.

## examples

● Inverse obstacle scattering problem:

$$
\begin{cases}
u = u^i + u^s, \;\; u^i = e^{ikx\cdot d} & \\
\Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\
u = 0, & \text{on } \partial\Omega \\
\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u^s}{\partial |x|} - iku^s\right) = 0 &
\end{cases}
$$

▶Determine the shape (and the location) of the domain, $\Omega$, from the far-field pattern (additional data), $u_\infty$. Where

$$u_\infty(\widehat{x}, d, k) = \lim_{|x|\to\infty} u^s(x, d)\sqrt{|x|}e^{-ik|x|}$$

- Fisher equation:

$$\begin{cases} \frac{d}{dt}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + au(x,t) - bu^2(x,t), \text{ on } (-1,1)\times(0,T] \\ u(-1,t) = u(1,t) = 0 \\ u(x,0) = u_0(x), \text{ on } (-1,1) \end{cases}$$

► Can we determine $D$, $a$, and $b$ from suitable additional conditions?

## 2. Reaction-Diffusion equations

$$\frac{d}{dt}u(x,t) - f(u) + \operatorname{div} J = 0$$

With $J = -D\nabla u$,

$$\frac{d}{dt}u(x,t) = \nabla \cdot (D\nabla u) + f(u)$$[a]

[a] [1]Turing(1952)

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## examples

- Fisher equation;

  $D$ is a constant, and $f(u)$ is the logistic model.

- Nagumo equation;

  $D$ is a constant, and $f(u)$ is a cubic.[a]

[a] [5], Murray or most of textbooks

- Fisher equation-Inhomogeneous growth conditions[a];

$$\frac{\partial}{\partial t}u(x,t) = D\Delta u(x,t) + U(x-vt)u(x,t) - bu(x,t)^2$$

where, $$\begin{cases} U(x-vt) = a, & |x-vt| \leq W/2 \\ U(x-vt) = -\varepsilon a, & |x-vt| > W/2 \end{cases}$$

Figure 1

[a] [2] Lin. et al.

- Fisher equation-The diffusion coefficient depends on the total population.[a]

$$\begin{cases} u_t = \frac{1}{D(P(t))} u_{xx} + au - bu^2, \ (x,t) \in (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x), \ x \in [0,1] \end{cases}$$

where, $P(t) = \int_0^1 u(x,t)dx$.

If $D(P) = P$, then a population is anxious to quickly move out of territories with low population densities.[b]

[a] [3] Ackleh

[b] see also [4] and reference in there

## 3. Fisher equation [a]

- Stationary population

$$\begin{cases} \frac{d}{dt}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + au(x,t) - bu^2(x,t) \\ u(-w,t) = u(w,t) = 0 \end{cases}$$

Set $\frac{du}{dt} = 0$, then

$$\begin{cases} \frac{d^2}{d\xi^2}u(\xi) + au(\xi) - bu^2(\xi) = 0 \\ u(\pm w/\sqrt{D}) = 0 \end{cases}$$

where, $\xi = x/\sqrt{D}$.

[a][6] Kenkre, Kuperman(2003)

- The elliptic solutions[a] of stationary population equation.

$$u(\xi) = \frac{a}{b}[f_1(k)cd^2(\sqrt{a}f_2(f)\xi, k) + f_3(k)]$$

where,

$$\begin{cases} f_1(k) = (3/2)k^2({k'}^2 + k^4)^{-1/2} \\ f_2(k) = (1/2)({k'}^2 + k^4)^{-1/4} \\ f_3(k) = (1/2)[1 - (k^2+1)({k'}^2 + k^4)^{-1/2}] \end{cases}$$

Here ${k'}^2 = 1 - k^2$

[a] see [7] for the elliptic function $cd$

- Recover $a, b$, and $k$ from the observed data by least squares method

$$\min_{a,b,k} \sum_i ||\phi(\xi_i) - u(a,b,k;\xi_i)||$$

where, $\phi(\xi_i)$ is the given data.

Figure 2

The data are plotted as circles and the full line curve shows the best fit.

- Applicability of the fisher equation.

Figure 3

In (a) the Fisher equation can be considered applicable while in (b) Nagumo equation cannot.

From the data $u(\xi_i)$, compute $u''(\xi_i)$ numerically.

Figure 4

second derivative of u is plotted against u in the two cases (a) and (b) of Fig. 3.

# References

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