Testing of Hypotheses

Definition

1. A (statistical) hypothesis is a statement about a population parameter.

2. The two complementary hypotheses in a testing problem are called the null hypothesis (denoted by \( H_0 \)) and the alternative hypothesis (denoted by \( H_1 \)).

Example: \( \theta = (\theta_1, \theta_2) \rightarrow \) average blood sugar level of a group of patients before and after taking a new drug

\( H_0 : \) no effect \( \leftrightarrow \) \( H_0 : \theta_1 = \theta_2 \)

\( H_1 : \) effective \( \leftrightarrow \) \( H_a : \theta_1 \neq \theta_2 \) or \( \theta_1 > \theta_2 \)

Definition: If a statistical hypothesis \( H \) completely specifies the distribution of \((X_1, \ldots, X_n)\), then it is simple; otherwise \( H \) is called composite.

Definition: Let \( \mathcal{X} \) be the set of all possible values of \( X = (X_1, \ldots, X_n) \). Then, a test function or a test rule \( \phi(X_1, \ldots, X_n) \) is a function from \( \mathcal{X} \) into \([0, 1]\) with the interpretation that if \( X = (X_1, \ldots, X_n) \) is observed then \( H_0 \) is rejected with probability \( \phi(X) \).

Definition: If \( \phi(X) \in \{0, 1\}, \forall X \in \mathcal{X} \), then

1. \( \phi(X) \) is called a simple test function (rule).

2. \( R_\phi = \{X : \phi(X) = 1\} \) is called the rejection region of \( \phi(X) \).

3. \( A_\phi = \{X : \phi(X) = 0\} \) is called the acceptance region of \( \phi(X) \).

Little Picture:
Note: Suppose $\phi(X)$ is a simple test rule for testing $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_0$. Then,

1. for any $\theta \in \Theta_0$, the probability of type I error at $\theta$

\[
P_\theta(\text{rejecting } H_0) = P_\theta(X = (X_1, \ldots, X_n) \in R_\phi) = E_\theta \phi(X),
\]

since $\phi(X) = \begin{cases} 1 & \text{if } X \in R_\phi \\ 0 & \text{if } X \not\in R_\phi \end{cases}$

2. and for any $\theta \not\in \Theta_0$, the probability of type II error at $\theta$ is

\[
P_\theta(\text{accepting } H_0) = P_\theta(X \in A_\phi) = 1 - P_\theta(X \in R_\phi) = 1 - E_\theta \phi(X)
\]

Remark: For any general test function $\phi(\cdot)$

1. Probability of a type I error at $\theta$ ($\theta \in \Theta_0$) = $E_\theta \phi(X)$ AND

2. Probability of a type II error at $\theta$ ($\theta \not\in \Theta_0$) = $1 - E_\theta \phi(X)$.

Definition: Let $\phi(X)$ be a test rule for testing $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \not\in \Theta_0$,

1. $\max_{\theta \in \Theta_0} E_\theta \phi(X)$ is called the size or the level of $\phi(X)$

2. $\Pi_\phi(\theta) = E_\theta \phi(X)$ is called the power function of $\phi(X)$.

Note: For $\theta \in \Theta_0$, $\Pi_\phi(\theta) = $ probability of type I error

For $\theta \not\in \Theta_0$, probability of type II error = $1 - \Pi_\phi(\theta)$