Asymptotic Efficiency

Recall: For two unbiased estimators $T$ and $T^*$ of $\gamma(\theta)$, we compare the variances of the estimators to judge their relative efficiency, i.e. $RE(T, T^*, \theta) = \frac{\text{Var}_\theta(T^*)}{\text{Var}_\theta(T)}$.

Similarly, we compare large-sample variances of asymptotically unbiased estimators.

Definitions: Let $\{T^*_n\}$ and $\{T_n\}$ be asymptotically unbiased for $\gamma(\theta)$. Then, define

1. The asymptotic relative efficiency of $\{T_n\}$ with respect to $\{T^*_n\}$ at $\theta$ is defined as
   
   $$ARE(T_n, T^*_n, \theta) = \lim_{n \to \infty} \frac{\text{Var}_\theta(T^*_n)}{\text{Var}_\theta(T_n)}, \quad \theta \in \Theta$$

2. $\{T^*_n\}$ is called asymptotically efficient if $ARE(T_n, T^*_n, \theta) \leq 1$, $\forall \theta \in \Theta$ and any other $\{T_n\}$ that is asymptotically unbiased for $\gamma(\theta)$.

3. The asymptotic efficiency of $\{T_n\}$ is defined as $AE(T_n, \theta) \equiv ARE(T_n, T^*_n, \theta)$ if $\{T^*_n\}$ is asymptotically efficient.

Previous example. Let $X_1, X_2, \ldots$ be iid uniform$(0, \theta)$, $\theta > 0$. Recall

- $T_n = \text{MME of } \theta \text{ based on } X_1, \ldots, X_n = 2\bar{X}_n$
- $T^*_n = \text{MLE of } \theta \text{ based on } X_1, \ldots, X_n = \max_{1 \leq i \leq n} X_i = X_{(n)}$
- $E_\theta T_n = \theta \quad \text{Var}_\theta(T_n) = \frac{3\theta^2}{n}$
- $E_\theta T^*_n = \frac{n}{n+1} \theta \quad \text{Var}_\theta(T^*_n) = \frac{n\theta^2}{(n+1)^2(n+2)}$
Asymptotic Properties of MLEs

Main Result\footnote{See Bickel and Doksum for details}: Let $X_1, X_2, \ldots, X_n$ be iid with common pmf/pdf $f(x|\theta)$. Let $\hat{\theta}_n \equiv \text{MLE of } \theta \in \Theta \subset \mathbb{R}$ based on $X_1, X_2, \ldots, X_n$. Then, under Cramér-Rao-type regularity conditions: as $n \to \infty$,

1. (Consistency) $\hat{\theta}_n \xrightarrow{p} \theta$

2. (Asymp. normality) $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N \left(0, \frac{1}{I_1(\theta)}\right)$, $I_1(\theta) = E_{\theta} \left(\frac{d \log f(X_1|\theta)}{d \theta} \right)^2$

3. (Asymp. efficiency) The sequence of estimators $\{\hat{\theta}_n\}$ is asymptotically efficient.

Discussion:

Remarks:

- The regularity conditions required for the validity of the Theorem above hold for distributions in the one-parameter exponential family: Binomial(*, $\theta$), Negative Binomial(*, $\theta$), Poisson($\theta$), $N(\ast, \theta)$, $N(\theta, \ast)$, Gamma(*, $\theta$), Gamma($\theta$, *), etc., where * indicates a given/known second parameter value.

- The following “Delta Method” result, combined with the Theorem above, is useful for finding the distribution of functions of $\hat{\theta}_n$, e.g., $g(\hat{\theta}_n)$. Because the result has wide applicability, we state it in a general form.

Delta Method: For a sequence $\{T_n\}$ of real-valued random variables, suppose it holds that $\sqrt{n}(T_n - a) \xrightarrow{d} N(0, \sigma^2(a))$, for some $a \in \mathbb{R}$. Then for any function $g : \mathbb{R} \to \mathbb{R}$ that is continuously differentiable at $a$ with derivative $g'(a) \neq 0$,

$$\sqrt{n}(g(T_n) - g(a)) \xrightarrow{d} N \left(0, [g'(a)]^2 \sigma^2(a) \right) \quad \text{as } n \to \infty$$
Example: Let $X_1, X_2, \ldots, X_n$ be iid Poisson($\theta$), $\theta > 0$.

1. Show that $T_n \equiv$ MLE of $\gamma(\theta) = \theta / (1 + \theta^2)$ is consistent.

2. Find the limiting distribution of $\sqrt{n}(T_n - \gamma(\theta))$. 